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IDEAL CONVERGENCE OF DOUBLE INTERVAL VALUED NUMBERS DEFINED BY ORLICZ FUNCTION

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ABSTRACT:

In this paper, we introduce some ideal convergent double interval valued numbers sequence spaces defined by Orlicz function and study different properties of these spaces like completeness, solidity, etc. We establish some inclusion relations among them.

Keywords: *Paranorm; completeness; ideal-convergence; interval numbers; Orlicz function.*

1. INTRODUCTION

The notion of I-convergence was initially introduced by Kostyrko, et. al [10] as a generalization of statistical convergence (see [8],[21]) which is based on the structure of the ideal I of subset of natural numbers $\mathbb{N}$. Kostyrko, et. al [11] gave some of basic properties of I-convergence and dealt with extremal I-limit points. Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X, here in our study it suffices to take I as a family of subsets of $\mathbb{N}$, positive integers, i.e. $I \subset 2^{\mathbb{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I.

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if $\Phi \notin F$, $A \cap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F. An ideal I is called *non-trivial* if $I \neq \Phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal I. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko, et.al (see [10]). Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number

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ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([10]). In this case we write $I - \lim x_k = \ell$. Further details on ideal convergence can be found in [20], [25]. The notion of I-convergence double sequence was initially introduced by Tripathy and Tripathy (see [24]).

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [14] in 1959 and Moore and Yang [15] in 1962. Further works on interval numbers can be found in Dwyer [3], Fischer [9] , Markov [13]. Furthermore, Moore and Yang [16], have developed applications to differential equations.

Chiao in [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Ideal of **N** and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Sengönül and Eryilmaz in [22] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Esi in [4], [5] introduced and studied strongly almost λ −convergence and statistically almost λ −convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. In [7], Esi and Hazarika introduced the difference classes of interval numbers. Recently Esi [6] has studied double sequences of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties.We denote the set of all real valued closed intervals by I$\mathbb{R}$. Any elements of I$\mathbb{R}$ is called closed interval and denoted by $\overline{x}$. That is $\overline{x} = \{x \in \mathbb{R} : a \leq x \leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers [1]. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in$I$\mathbb{R}$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x2l \leq x \leq x1r + x2r$,and if $\alpha \geq 0$, then $\alpha x = x \in \mathbb{R} : \alpha x1l \leq x \leq \alpha x1r$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{x}_1 . \overline{x}_2 = \left\{ \begin{matrix} x \in \mathbb{R} : \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \\ \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \end{matrix} \right\}.$$

In [14], Moore proved that the set of all interval numbers I$\mathbb{R}$ is a complete metric space defined by

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$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f: \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}, \overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [1].

Now we give the definition of convergence of interval numbers:

A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$ [1] .

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

Recall in [17],[12] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle [19]. An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$. Subsequently, the notion of Orlicz function was used to defined sequence spaces by Tripathy et al [23], Tripathy and Hazarika[26] and many others.

An interval valued double sequence $\overline{x} = (\overline{x}_{k,l})$ is said to be convergent in the Pringsheim's sense or P-convergent to an interval number $\overline{x}_o$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(\overline{x}_{k,l}, \overline{x}_o) < \varepsilon \text{for} k, l > N \quad (see\ Pringsheim)$$

and we denote it by $P - \lim \overline{x}_{k,l} = \overline{x}_o$, where $d\left(\overline{x}_{k,l}, \overline{y}_{k,l}\right)$ is the Hausdorff distance between $\overline{x} = (\overline{x}_{k,l})$ and $\overline{y} = \left(\overline{y}_{k,l}\right)$. The interval number $\overline{x}_o$ is called the Pringsheim limit of $\overline{x} = (\overline{x}_{k,l})$. More exactly, we say that a double sequence of interval numbers $\overline{x} = (\overline{x}_{k,l})$ converges to a finite interval number $\overline{x}_o$ if $\overline{x}_{k,l}$ tends to $\overline{x}_o$ as both k and l tend to infinity independently of each another. We denote by $\overline{c}^2$ the set of all double convergent interval numbers of double interval numbers.

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The interval number double sequence $\overline{x} = (\overline{x}_{k,l})$ is bounded if and only if there exists a positive number B such that $d(\overline{x}_{k,l}, \overline{0}) < B$ for all k and l. We shall denote all bounded interval number double sequences by $\overline{l}_{\infty}^{2}$. Let $\overline{w}^2$ denote the set of all double sequences of interval numbers.

Let $p = (p_{i,j})$ be a double sequence of positive real numbers. If $0 < p_{i,j} \le \sup_{i,j} p_{i,j} = H < \infty$ and $D = \max(1, 2^{H-1})$, then for $a_{i,j}, b_{i,j} \in \mathbb{R}$ for all $i,j \in \mathbb{N}$, we have

$$|a_{i,j} + b_{i,j}|^{p_{i,j}} \le D(|a_{i,j}|^{p_{i,j}} + |b_{i,j}|^{p_{i,j}}).$$

2. MAIN RESULTS

In this paper, we define new double sequence spaces for interval sequences as follows.

Let $\mathcal{J}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. Let M be an Orlicz function and $p = (p_{i,j})$ be a double sequence of strictly positive real numbers. We introduce the following sequence spaces:

$$_2\overline{w}^{\mathcal{J}}(M,p) = \left\{ \begin{array}{l} \overline{x} = (\overline{x}_{i,j}) : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j}, \overline{x}_o)}{\rho} \right) \right]^{p_{i,j}} \ge \varepsilon \right\} \in \mathcal{J}, \\ for\ some \rho > 0,\ and \overline{x}_o \in \mathbf{IR} \end{array} \right\},$$

$$_2\overline{w}_o^{\mathcal{J}}(M,p) = \left\{ \begin{array}{l} \overline{x} = (\overline{x}_{i,j}) : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j}, \overline{0})}{\rho} \right) \right]^{p_{i,j}} \ge \varepsilon \right\} \in \mathcal{J}, \\ for\ some \rho > 0 \end{array} \right\}$$

$$_2\overline{w}_{\infty}^{\mathcal{J}}(M,p) = \left\{ \begin{array}{l} \overline{x} = (\overline{x}_{i,j}) : \exists K > 0 s.t. \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j}, \overline{0})}{\rho} \right) \right]^{p_{i,j}} \ge K \right\} \in \mathcal{J}, \\ for\ some \rho > 0 \end{array} \right\}.$$

and

$$_2\overline{w}_{\infty}(M,p) = \left\{ \begin{array}{l} \overline{x} = (\overline{x}_{i,j}) : \sup_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j}, \overline{0})}{\rho} \right) \right]^{p_{i,j}} < \infty, \\ for\ some \rho > 0 \end{array} \right\}.$$

Theorem 2.1. *Let $p = (p_{i,j})$ be bounded. Then the double sequence spaces $_2\overline{w}^{\mathcal{J}}(M,p)$, $_2\overline{w}_o^{\mathcal{J}}(M,p)$ and $_2\overline{w}_\infty^{\mathcal{J}}(M,p)$ are linear spaces.*

Proof. It is easy, so omitted it.

Theorem 2.2.*The double sequence spaces $_2\overline{w}^{\mathcal{J}}(M,p)$, $_2\overline{w}_o^{\mathcal{J}}(M,p)$ and $_2\overline{w}_\infty^{\mathcal{J}}(M,p)$ are paranormed sequence spaces paranormed by*

$$\overline{g}(\overline{x}) = \inf\left\{\rho^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{\rho}\right) \leq 1\right\}$$

where $H = max\left(1, \sup_{i,j} p_{i,j} < \infty\right)$.

Proof. Clearly $\overline{g}(\overline{0}) = 0, \overline{g}(\overline{x}) = \overline{g}(-\overline{x})$. Let $\overline{x} = (\overline{x}_{i,j}), \overline{y} = (\overline{y}_{i,j}) \in {}_2\overline{w}^{\mathcal{J}}(M,p)$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_{i,j} M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{\rho_1}\right) \leq 1 \text{and} \sup_{i,j} M\left(\frac{d(\overline{y}_{i,j},\overline{0})}{\rho_2}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_{i,j} M\left(\frac{d(\overline{x}_{i,j}+\overline{y}_{i,j},\overline{0})}{\rho}\right)$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2} \sup_{i,j} M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} \sup_{i,j} M\left(\frac{d(\overline{y}_{i,j},\overline{0})}{\rho_2}\right)$$

$$\leq 1.$$

Now

$$\overline{g}(\overline{x}+\overline{y}) = \inf\left\{(\rho_1+\rho_2)^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\overline{x}_{i,j}+\overline{y}_{i,j},\overline{0})}{\rho}\right) \leq 1\right\}$$

$$\leq \inf\left\{\rho_1^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{\rho_1}\right) \leq 1\right\}$$

$$+\inf\left\{\rho_2^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\overline{y}_{i,j},\overline{0})}{\rho_2}\right) \leq 1\right\}$$

$$= \overline{g}(\overline{x}) + \overline{g}(\overline{y}).$$

Let $\beta \in \mathbb{R}$, then the continuity of the product follows from the following inequality:

$$\overline{g}(\beta\overline{x}) = \inf\left\{\rho^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\beta\overline{x}_{i,j},\overline{0})}{\rho}\right) \le 1\right\}$$

$$= \inf\left\{(|\beta|r)^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{r}\right) \le 1\right\}$$

where $\frac{1}{r} = \frac{|\beta|}{\rho}$. This completes the proof.

Theorem 2.3. *The double sequence spaces* $_2\overline{w}^{\mathcal{I}}(M,p)$, $_2\overline{w}_o^{\mathcal{I}}(M,p)$, $_2\overline{w}_\infty^{\mathcal{I}}(M,p)$ *and* $_2\overline{w}_\infty(M,p)$ *are complete paranormed spaces, paranormed by* $\overline{g}$ *defined by Theorem 2.2.*

Proof. Let $(\overline{x}_{i,j}^s)$ be a Cauchy sequence in $_2\overline{w}_\infty(M,p)$. Then $\overline{g}\left((\overline{x}_{i,j}^s) - (\overline{x}_{i,j}^t)\right) \to 0$ as $s,t \to \infty$. For given $\varepsilon > 0$, choose $r > 0$ and $x_o > 0$ be such that $\frac{\varepsilon}{rx_o} > 0$ and $M\left(\frac{rx_o}{2}\right) \ge 1$. Now $\overline{g}\left((\overline{x}_{i,j}^s) - (\overline{x}_{i,j}^t)\right) \to 0$ as $s,t \to \infty$ implies that there exists $n_o \in \mathbb{N}$ such that

$$\overline{g}\left((\overline{x}_{i,j}^s) - (\overline{x}_{i,j}^t)\right) < \frac{\varepsilon}{rx_o} \text{ for all } s,t \ge n_o.$$

Then

$$\inf\left\{\rho^{\frac{p_{i,j}}{H}} : \sup_{i,j} M\left(\frac{d\left(\overline{x}_{i,j}^s - \overline{x}_{i,j}^t,\overline{0}\right)}{\rho}\right) \le 1\right\} < \frac{\varepsilon}{rx_o}. \tag{2.1}$$

Now from (2.1), we have

$$M\left(\frac{d\left(\overline{x}_{i,j}^s - \overline{x}_{i,j}^t,\overline{0}\right)}{\rho}\right) \le 1 \le M\left(\frac{rx_o}{2}\right)$$

$$\Rightarrow \frac{d\left(\overline{x}_{i,j}^s - \overline{x}_{i,j}^t,\overline{0}\right)}{\overline{g}\left((\overline{x}_{i,j}^s) - (\overline{x}_{i,j}^t)\right)} < \frac{rx_o}{2}.\frac{\varepsilon}{rx_o} = \frac{\varepsilon}{2}.$$

This implies that $(\overline{x}_{i,j}^s)$ is a Cauchy sequence of real numbers. Let $\lim_{s\to\infty}\overline{x}_{i,j}^s = \overline{x}_{i,j}$. Using continuity of M, we have

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$$\lim_{t\to\infty}\sup_{i,j} M\left(\frac{d\left(\overline{x}_{i,j}^{s}-\overline{x}_{i,j}^{t},\overline{0}\right)}{\rho}\right) \le 1$$

$$\Rightarrow \sup_{i,j} M\left(\frac{d\left(\overline{x}_{i,j}^{s}-\overline{x}_{i,j},\overline{0}\right)}{\rho}\right) \le 1.$$

Let $s \ge n_o$, then taking infimum of such $\rho's$, we have $\overline{g}\left(\left(\overline{x}_{i,j}^{s}\right)-\left(\overline{x}_{i,j}\right)\right) < \varepsilon$. Thus $\left(\overline{x}_{i,j}^{s}\right)-\left(\overline{x}_{i,j}\right) \in {}_2\overline{w}_\infty(M,p)$. By linearity of the double space ${}_2\overline{w}_\infty(M,p)$, we have $\left(\overline{x}_{i,j}\right) \in {}_2\overline{w}_\infty(M,p)$. Hence ${}_2\overline{w}_\infty(M,p)$ is complete. This completes the proof.

Theorem 2.4. (a) ${}_2\overline{w}^{\mathcal{I}}(M,p) \subset {}_2\overline{w}_\infty(M,p)$,

(b) ${}_2\overline{w}_o^{\mathcal{I}}(M,p) \subset {}_2\overline{w}_\infty(M,p)$.

Proof. It is easy, so omitted.

Theorem 2.5. *The double sequence spaces* ${}_2\overline{w}^{\mathcal{I}}(M,p)$ *and* ${}_2\overline{w}_o^{\mathcal{I}}(M,p)$ *are nowhere dense subsets of* ${}_2\overline{w}_\infty(M,p)$.

Proof. The proof is obvious in view of Theorem 2.3 and Theorem 2.4.

Theorem 2.6. (a) *If* $0 < inf_{i,j}p_{i,j} \le p_{i,j} < 1$, *then* ${}_2\overline{w}^{\mathcal{I}}(M,p) \subset {}_2\overline{w}^{\mathcal{I}}(M)$,

(b) *If* $1 < p_{i,j} < sup_{i,j}p_{i,j} < \infty$, *then* ${}_2\overline{w}^{\mathcal{I}}(M) \subset {}_2\overline{w}^{\mathcal{I}}(M,p)$,

(c) *If* $0 < p_{i,j} \le q_{i,j} < \infty$ *and* $\left(\frac{q_{i,j}}{p_{i,j}}\right)$ *is bounded, then* ${}_2\overline{w}^{\mathcal{I}}(M,p) \subset {}_2\overline{w}^{\mathcal{I}}(M,q)$.

Proof. The first part of the result follows from the relation

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} M\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right) \ge \varepsilon\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} \ge \varepsilon\right\}$$

and the second part of the result follows from the relation

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \left[M\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} \ge \varepsilon\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} M\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right) \geq \varepsilon\right\}.$$

This completes the proof.

The proof of the part three is easy, so omitted.

Theorem 2.7.(a) *If* $0 < inf_{i,j} p_{i,j} \leq p_{i,j} < 1$, *then* ${}_2\overline{w}_0^{\mathcal{J}}(M,p) \subset {}_2\overline{w}_0^{\mathcal{J}}(M)$,

(b) *If* $1 < p_{i,j} < sup_{i,j} p_{i,j} < \infty$, *then* ${}_2\overline{w}_0^{\mathcal{J}}(M) \subset {}_2\overline{w}_0^{\mathcal{J}}(M,p)$,

(c) *If* $0 < p_{i,j} \leq q_{i,j} < \infty$ *and* $\left(\frac{q_{i,j}}{p_{i,j}}\right)$ *is bounded, then* ${}_2\overline{w}_0^{\mathcal{J}}(M,p) \subset {}_2\overline{w}_0^{\mathcal{J}}(M,q)$.

Proof of the result follows from the Theorem 2.6.

Theorem 2.8. *Let* M_1 *and* M_2 *be two Orlicz functions. Then*

$${}_2\overline{w}^{\mathcal{J}}(M_1,p) \cap {}_2\overline{w}^{\mathcal{J}}(M_2,p) \subset {}_2\overline{w}^{\mathcal{J}}(M_1+M_2,p).$$

Proof. Let $(\overline{x}_{i,j}) \in {}_2\overline{w}^{\mathcal{J}}(M_1,p) \cap {}_2\overline{w}^{\mathcal{J}}(M_2,p)$. Then for every $\varepsilon > 0$ we have

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho_1}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{J}, \quad for some \rho_1 > 0$$

and

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M_2\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho_2}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{J}, \quad for some \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the following inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[(M_1+M_2)\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}}$$

$$\leq D\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho_1}\right)\right]^{p_{i,j}} + \sum_{i=1}^{m} \sum_{j=1}^{n} \left[M_2\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho_2}\right)\right]^{p_{i,j}}\right).$$

This completes the proof.

Theorem 2.9. *Let M_1 and M_2 be two Orlicz functions. Then*

$$ {}_2\overline{w}^{\mathcal{I}}(M_1,p) \subset {}_2\overline{w}^{\mathcal{I}}(M_2 \circ M_1,p). $$

Proof. Let $\inf p_{i,j} = H_0$. For given $\varepsilon > 0$, we first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of M_2 choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M_2(t) < \varepsilon_0$. Let $(\overline{x}_{i,j}) \in {}_2\overline{w}^{\mathcal{I}}(M_1,p)$. Now from the definition of ${}_2\overline{w}^{\mathcal{I}}(M_1,p)$, for some $\rho > 0$

$$ \overline{A}(\delta) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} \geq \delta^H\right\} \in \mathcal{I}. $$

Thus if $(m,n) \notin \overline{A}(\delta)$ then we have

$$ \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} < \delta^H $$

$$ \Rightarrow \sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} < mn\delta^H $$

$$ \Rightarrow \left[M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right]^{p_{i,j}} < \delta^H, for all i,j = 1,2,3 \ldots $$

$$ \Rightarrow M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right) < \delta, for all i,j = 1,2,3 \ldots. $$

Hence from above inequality and using continuity of M_2, we must have

$$ M_2\left(M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right) < \varepsilon_0, for all i,j = 1,2,3 \ldots. $$

which consequently implies that

$$ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_2\left(M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right)\right]^{p_{i,j}} < mn\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < mn\varepsilon $$

$$ \Rightarrow \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_2\left(M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right)\right]^{p_{i,j}} < \varepsilon. $$

This shows that

$$ \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M_2\left(M_1\left(\frac{d(\overline{x}_{i,j},\overline{x}_o)}{\rho}\right)\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \subset \overline{A}(\delta) $$

and so belongs to $\mathcal{I}$. This completes the proof.

Theorem 2.10. *Let M_1 and M_2 be two Orlicz functions. Then*

(a) ${}_2\overline{w}_0^{\mathcal{I}}(M_1,p) \cap {}_2\overline{w}_0^{\mathcal{I}}(M_2,p) \subset {}_2\overline{w}_0^{\mathcal{I}}(M_1+M_2,p)$;

(b) ${}_2\overline{w}_0^{\mathcal{I}}(M_1,p) \subset {}_2\overline{w}_0^{\mathcal{I}}(M_2 \circ M_1,p)$.

The proof of the theorem follows from the Theorems 2.8 and 2.9.

Theorem 2.11. *Let M_1 and M_2 be two Orlicz functions satisfying Δ_2-condition. If $\beta = lim_{t\to\infty}\frac{M_2(t)}{t} \geq 1$, then*

(a) ${}_2\overline{w}_0^{\mathcal{I}}(M_1,p) = {}_2\overline{w}_0^{\mathcal{I}}(M_2 \circ M_1,p)$,

(b) ${}_2\overline{w}^{\mathcal{I}}(M_1,p) = {}_2\overline{w}^{\mathcal{I}}(M_2 \circ M_1,p)$.

Proof. It is easy, so omitted.

Theorem 2.12. *The double sequence space ${}_2\overline{w}^{\mathcal{I}}(M,p)$, ${}_2\overline{w}_o^{\mathcal{I}}(M,p)$, ${}_2\overline{w}_\infty^{\mathcal{I}}(M,p)$ and ${}_2\overline{w}_\infty(M,p)$ are solid as well as monotone.*

Proof. We give the proof for only ${}_2\overline{w}_o^{\mathcal{I}}(M,p)$. The others can be proved similarly. Let $\overline{x} = (\overline{x}_{i,j}) \in {}_2\overline{w}_o^{\mathcal{I}}(M,p)$ and $(\alpha_{i,j})$ be a scalar sequence such that $|\alpha_{i,j}| \leq 1$ for all $i,j \in \mathbb{N}$. Then for every $\varepsilon > 0$ we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M\left(\frac{d(\alpha_{i,j}\overline{x}_{i,j},\overline{0})}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{E}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[M\left(\frac{d(\overline{x}_{i,j},\overline{0})}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{I},$$

where $E = \max\{1, |\alpha_{k,l}|^H\}$. Hence $(\alpha\overline{x}) \in {}_2\overline{w}_o^{\mathcal{I}}(M,p)$. This completes the proof.

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