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Research Article

A Study On the Sums of Squares of Generalized Tribonacci Numbers: Closed Form Formulas of $\sum_{k=0}^{n} kx^k W_k^2$

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Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^k W_k^2$, $\sum_{k=0}^{n} kx^k W_{k+2}W_k$ and $\sum_{k=0}^{n} kx^k W_{k+1}W_k$ for the squares of generalized Tribonacci numbers are presented. Here, $\{W_m\}_{m\in\mathbb{Z}}$ is the generalized Tribonacci sequence, n is a non-negative integer and x is a real or complex number. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations.

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Keywords. Sums of squares, third order recurrence, generalized Tribonacci numbers, Padovan numbers, Perrin numbers, Narayana numbers.

1. Introduction

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1,2,6,7,13,14,19,20,21,23,35,36,37,38].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1 A few special case of generalized Tribonacci sequences.

Sequences (Numbers)	Notation	OEIS [22]
Tribonacci	$\{T_n\} = \{V_n(0,1,1;1,1,1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{V_n(3,1,3;1,1,1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{V_n(0,1,2;2,1,1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{V_n(3,2,6;2,1,1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{V_n(0,1,1;2,1,1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{V_n(1,1,1;0,1,1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{V_n(3,0,2;0,1,1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{V_n(0,0,1;0,1,1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{V_n(1,1,1;0,2,1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{V_n(3,0,2;0,2,1)\}$	
Jacobsthal-Padovan	$\{Q_n\} = \{V_n(1,1,1;0,1,2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{L_n\} = \{V_n(3,0,2;0,1,2)\}$	A072328
Narayana	$\{N_n\} = \{V_n(0,1,1;1,0,1)\}$	A078012
Narayana-Lucas	$\{U_n\} = \{V_n(3,1,1;1,0,1)\}$	A001609
Narayana-Perrin	$\{H_n\} = \{V_n(3,0,2;1,0,1)\}$	
third order Jacobsthal	$\{J_n^{(3)}\} = \{V_n(0,1,1;1,1,2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{V_n(2,1,5;1,1,2)\}$	A226308
3-primes	$\{G_n\} = \{V_n(0,1,2;2,3,5)\}$	
Lucas 3-primes	$\{H_n\} = \{V_n(3,2,10;2,3,5)\}$	
modified 3-primes	$\{E_n\} = \{V_n(0,1,1;2,3,5)\}$	

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$\sum_{k=0}^{n} k(-1)^k T_k^2 = \frac{1}{4}((-1)^n\left((n+1)T_{n+3}^2-(2n+1)T_{n+2}^2+(3n+2)T_{n+1}^2-2(n+2)T_{n+1}T_{n+3}+2T_{n+2}T_{n+1}\right)+1)$

and

$\sum_{k=0}^{n} k(-1)^k N_k^2 = \frac{1}{9}((-1)^n\left((3n+7)N_{n+3}^2-(6n+5)N_{n+2}^2+(6n-1)N_{n+1}^2-6N_{n+3}N_{n+2}-2(3n+7)N_{n+3}N_{n+1}+2(3n+10)N_{n+2}N_{n+1}\right)-1)$.

In this work, we derive expressions for sums of second powers of generalized Tribonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[3,4,10,11,12,24,33,30]	[9,25,27,28,31,32,39]	[5,8,15]
Generalized Tribonacci	[17,26,29]		
Generalized Tetranacci	[16,18,34]		

Let

$$\Delta = (-t^2x^3+sx+rtx^2+1)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1).$$

Theorem 1.1. *If* $\Delta \neq 0$ *then*

(a):

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{\Delta}$$

(b):

$$\sum_{k=0}^{n} x^k W_{k+1} W_k = \frac{\Delta_2}{\Delta}$$

(c):

$$\sum_{k=0}^{n} x^k W_{k+2} W_k = \frac{\Delta_3}{\Delta},$$

where

$\Delta_1 = -x^{n+3}(t^2x^3+sx+rtx^2-1)W_{n+3}^2 - x^{n+2}(r^2x+t^2x^3+sx+r^2t^2x^4+rtx^2+r^2sx^2+r^3tx^3+2rstx^3-1)$ $W_{n+2}^2 - x^{n+1}(r^2x+s^2x^2-s^3x^3+t^2x^3+sx+r^2t^2x^4+s^2t^2x^5+rtx^2+r^2sx^2+r^3tx^3+4rstx^3-rs^2tx^4-1)$ $W_{n+1}^2 + x^2(t^2x^3+sx+rtx^2-1)W_2^2 + x(r^2x+t^2x^3+sx+r^2t^2x^4+rtx^2+r^2sx^2+r^3tx^3+2rstx^3-1)$ $W_1^2 + (r^2x+s^2x^2-s^3x^3+t^2x^3+sx+r^2t^2x^4+s^2t^2x^5+rtx^2+r^2sx^2+r^3tx^3+4rstx^3-rs^2tx^4-1)$ $W_0^2 + 2x^{n+4}(r+tx)(s+rtx)W_{n+3}W_{n+2} + 2x^{n+4}t(r+stx^2)W_{n+3}W_{n+1} - 2x^{n+4}t(sx-1)(s+rtx)W_{n+2}W_{n+1} -$ $2x^3(r+tx)(s+rtx)W_2W_1 - 2tx^3(r+stx^2)W_2W_0 + 2x^3t(sx-1)(s+rtx)W_1W_0$

and

$\Delta_2 = x^{n+3}(r+stx^2)W_{n+3}^2 + x^{n+4}(t+rs)(s+rtx)W_{n+2}^2 + x^{n+4}t^2(r+stx^2)W_{n+1}^2 - x^{n+2}(r^2x+s^2x^2+$ $t^2x^3+2rstx^3-1)W_{n+3}W_{n+2} + x^{n+3}t(r^2x-s^2x^2-t^2x^3+1)W_{n+3}W_{n+1} - x^{n+1}(r^2x+s^2x^2-s^3x^3+t^2x^3+$ $sx+rtx^2+r^2sx^2+r^3tx^3-rt^3x^5-st^2x^4+2rstx^3-rs^2tx^4-1)W_{n+2}W_{n+1} - x^2(r+stx^2)W_2^2 - x^3(t+$ $rs)(s+rtx)W_1^2 - x^3t^2(r+stx^2)W_0^2 + x(r^2x+s^2x^2+t^2x^3+2rstx^3-1)W_2W_1 - x^2t(r^2x-s^2x^2-t^2x^3+$ $1)W_2W_0 + (r^2x+s^2x^2-s^3x^3+t^2x^3+sx+rtx^2+r^2sx^2+r^3tx^3-rt^3x^5-st^2x^4+2rstx^3-rs^2tx^4-1)$ W_1W_0

and

$\Delta_3 = x^{n+3}(s-s^2x+r^2+rtx)W_{n+3}^2 + x^{n+2}(s-s^2x+r^2t^2x^3-r^2sx+rt^3x^4-rs^2tx^3)W_{n+2}^2 + x^{n+4}t^2(s-s^2x+$ $r^2+rtx)W_{n+1}^2 - x^{n+2}(r+tx)(r^2x-s^2x^2+t^2x^3-1)W_{n+3}W_{n+2} - x^{n+1}(r^2x+s^2x^2-s^3x^3+t^2x^3+sx+r^2sx^2-$ $st^2x^4+2rstx^3-1)W_{n+3}W_{n+1} + x^{n+2}t(sx-1)(r^2x-s^2x^2+t^2x^3-1)W_{n+2}W_{n+1} - x^2(s-s^2x+r^2+rtx)W_2^2+$ $x(-s+s^2x-r^2t^2x^3+r^2sx-rt^3x^4+rs^2tx^3)W_1^2 - x^3t^2(s-s^2x+r^2+rtx)W_0^2 + x(r+tx)(r^2x-s^2x^2+t^2x^3-$ $1)W_2W_1 + (r^2x+s^2x^2-s^3x^3+t^2x^3+sx+r^2sx^2-st^2x^4+2rstx^3-1)W_2W_0 - xt(sx-1)(r^2x-s^2x^2+t^2x^3-1)$ $W_1W_0.$ 3

Proof. The proof is given in [29, Theorem 3.1.].

2. Main Result

Let

$$\Omega = (-t^2x^3+sx+rtx^2+1)^2(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)^2$$

Theorem 2.1. *Let x be a real or complex number. If $\Omega \neq 0$ then*

(a):

$$\sum_{k=0}^{n} kx^k W_k^2 = \frac{\Omega_1}{\Omega},$$

(b):

$$\sum_{k=0}^{n} kx^k W_{k+1} W_k = \frac{\Omega_2}{\Omega},$$

(c):

$$\sum_{k=0}^{n} kx^k W_{k+2} W_k = \frac{\Omega_3}{\Omega},$$

where

$$\Omega_1 = \sum_{k=1}^{12} \Gamma_k, \quad \Omega_2 = \sum_{k=1}^{12} \Phi_k, \quad \Omega_3 = \sum_{k=1}^{12} \Psi_k,$$

with

$\Gamma_1 = x^{n+3}(n(t^2x^3 - sx - rtx^2 - 1)(t^2x^3 + sx + rtx^2 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 2r^2x - 2s^2x^2 - 2s^3x^3 + 6t^2x^3 + s^4x^4 - 3t^4x^6 + 6sx - 2r^2s^2x^3 - 9r^2t^2x^4 - 2r^4t^2x^5 - 6s^2t^2x^5 - 4r^3t^3x^6 - 2r^2t^4x^7 + 4s^3t^2 x^6 - s^2t^4x^8 + 6rtx^2 - 6rt^3x^5 - 6st^2x^4 - 12r^2st^2x^5 + 2rs^2t^3x^7 - 6rs^2tx^4 - 4r^3stx^4 + 2rs^3tx^5 - 12rst^3x^6 - 3 - 2r^2sx^2 - 6rstx^3 + r^2s^2t^2x^6 - 4r^3tx^3)W_{n+3}^2,$

$\Gamma_2 = x^{n+2}(n(r^2x + t^2x^3 + sx + r^2t^2x^4 + rtx^2 + r^2sx^2 + r^3tx^3 + 2rstx^3 - 1)(t^2x^3 - sx - rtx^2 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 4r^2x - 2r^4x^2 - 2s^2x^2 + 3t^2x^3 - t^6x^9 + 4sx - 5r^2s^2x^3 - 2r^2s^3x^4 - 2r^2t^2x^4 - 2r^4 s^2x^4 + r^2s^4x^5 - 8r^4t^2x^5 - 7s^2t^2x^5 - 12r^3t^3x^6 - 11r^2t^4x^7 + 4s^3t^2x^6 - 2r^6t^2x^6 - 4r^5t^3x^7 - 2r^4t^4x^8 - 4r^4s x^3 - 2rt^3x^5 - 2st^2x^4 - 4r^5tx^4 - 2rt^5x^8 - 2st^4x^7 - 20r^2st^2x^5 - 14r^3s^2tx^5 + 4rs^2t^3x^7 + 2r^3s^3tx^6 - 16r^4st^2x^6 - 14r^3st^3x^7 + 2rstx^3 - 18r^2s^2t^2x^6 + 6r^2s^3t^2x^7 + r^4s^2t^2x^7 + 2r^3s^2t^3x^8 - r^2s^2t^4x^9 - 8rs^2tx^4 - 16r^3stx^4 - 6 rs^3tx^5 - 18rst^3x^6 + 4rs^4tx^6 - 4r^5stx^5 - 2rst^5x^9 - 2 + 4rtx^2)W_{n+2}^2,$

$\Gamma_3 = x^{n+1}(n(t^2x^3 - sx - rtx^2 - 1)(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + r^2t^2x^4 + s^2t^2x^5 + rtx^2 + r^2sx^2 + r^3tx^3 + 4rstx^3 - rs^2tx^4 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 2r^2x - r^4x^2 + s^2x^2 - 4s^3x^3 + s^4x^4 + 2s^5x^5 - s^6x^6 + 3t^4x^6 - 2t^6x^9 + 2sx - 4r^2s^2x^3 - r^4s^2x^4 + 2r^2s^4x^5 - 4r^4t^2x^5 - 2s^2t^2x^5 - 8r^3t^3x^6 - 10r^2t^4x^7 - r^6t^2x^6 - 2r^5t^3x^7 - 2s^4t^2x^7 - 2r^4t^4x^8 - 8s^2t^4x^8 - 2r^3t^5x^9 + 2s^5t^2x^8 - r^2t^6x^{10} + 4s^3t^4x^9 - s^4t^4x^{10} + 2rt^3x^5 + 2st^2x^4 - 2r^5t x^4 - 4rt^5x^8 - 4st^4x^7 - 14r^2st^2x^5 - 8r^3s^2tx^5 + 4r^3s^3tx^6 - 10r^4st^2x^6 - 10r^3st^3x^7 - 8rs^3t^3x^8 - 6r^2st^4x^8 + 2rs^4t^3x^9 + 2rs^2t^5x^{10} + 6rstx^3 - 19r^2s^2t^2x^6 + 8r^2s^3t^2x^7 + 2r^4s^2t^2x^7 - r^2s^4t^2x^8 - 2r^2s^2t^4x^9 - 12rs^2tx^4 - 12r^3stx^4 - 4rs^3tx^5 - 16rst^3x^6 + 10rs^4tx^6 - 2r^5stx^5 - 2rs^5tx^7 - 8rst^5x^9 - 1 + 2rtx^2 - 2r^4sx^3)W_{n+1}^2,$ 4

$\Gamma_4 = 2x^{n+4}(n(r + tx)(-t^2x^3 + sx + rtx^2 + 1)(s + rtx)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 2r^3s^2x^2 - 4rs + 8r^3t^2x^3 + 8r^2t^3x^4 + 2r^5t^2x^4 + 4r^4t^3x^5 - s^2t^3x^5 + 2r^3t^4x^6 + 3rs^2x + 3r^3sx - 5r^2tx + 2rs^3x^2 - 6rt^2x^2 - rs^4 x^3 + 4r^4tx^2 + 4s^2tx^2 + 3s^3tx^3 + 6rt^4x^5 + 4st^3x^4 - 2s^4tx^4 + st^5x^7 - 5stx + 10r^2s^2tx^3 + 12rs^2t^2x^4 - 2r^2s^3tx^4 + 14r^3st^2x^4 - 5rs^3t^2x^5 + 13r^2st^3x^5 + rs^2t^4x^7 - r^3s^2t^2x^5 - 2r^2s^2t^3x^6 + 10r^2stx^2 + 10rst^2x^3 + 4r^4stx^3)W_{n+3}W_{n+2},$

$\Gamma_5 = 2tx^{n+4}(n(-t^2x^3 + sx + rtx^2 + 1)(r + stx^2)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 3r^3x - 4r + r^2t^3x^5 - 2s^2t^3x^6 + s^3t^3x^7 - 6stx^2 + 2rs^2x^2 + 2r^3sx^2 - rs^3x^3 + 2r^2tx^2 + 2rt^2x^3 + r^4tx^3 + 5s^2tx^3 + 4 s^3tx^4 + 2rt^4x^6 + 6st^3x^5 - 3s^4tx^5 + 3rsx + 11rs^2t^2x^5 + 3r^3st^2x^5 - 2rs^3t^2x^6 + 2r^2st^3x^6 + 9r^2stx^3 + 4rst^2x^4 - rst^4 x^7 + 4r^2s^2tx^4)W_{n+3}W_{n+1},$

$\Gamma_6 = 2tx^{n+4}(n(sx - 1)(s + rtx)(t^2x^3 - sx - rtx^2 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 8s^2x - 4s - 2s^3x^2 - 4s^4x^3 + 2s^5x^4 - 2r^2s^2x^2 - 3r^2s^3x^3 + 3r^2t^2x^3 + 2r^4t^2x^4 - 4s^2t^2x^4 + r^3t^3x^5 - s^2t^4x^7 + 4r^3tx^2 + 4rt^3x^4 + 2 st^2x^3 + rt^5x^7 + 2st^4x^6 - 5rtx + 4r^2st^2x^4 - 6r^3s^2tx^4 + 2rs^2t^3x^6 - 3r^4st^2x^5 - 2r^3st^3x^6 - rs^3t^3x^7 + r^2st^4x^7 + 12rstx^2 - 14r^2s^2t^2x^5 + 2r^2s^3t^2x^6 - rs^2tx^3 - r^3stx^3 - 14rs^3tx^4 - 6rst^3x^5 + 4rs^4tx^5 + 3r^2sx)W_{n+2}W_{n+1},$

$\Gamma_7 = x^2(-r^2x + 2s^2x^2 - 3t^2x^3 + t^6x^9 - 4sx + r^2s^2x^3 + 8r^2t^2x^4 + r^4t^2x^5 + 7s^2t^2x^5 + 2r^3t^3x^6 + 2r^2t^4x^7 - 4s^3t^2x^6 - 4rtx^2 + 2r^2sx^2 + 4r^3tx^3 + 2rt^3x^5 + 2st^2x^4 + 2rt^5x^8 + 2st^4x^7 + 6r^2st^2x^5 - 2rs^2t^3x^7 + 8rstx^3 + 2r^3stx^4 + 10rst^3x^6 + 2)W_2^2,$

$\Gamma_8 = x(-2r^2x + r^4x^2 + 2s^2x^2 - 2s^3x^3 + s^4x^4 - 3t^4x^6 + 2t^6x^9 - 2sx + 2r^2s^2x^3 + 2r^2s^3x^4 + r^4s^2x^4 + 4r^4t^2x^5 + 8s^2t^2x^5 + 8r^3t^3x^6 + 10r^2t^4x^7 - 4s^3t^2x^6 + r^6t^2x^6 + 2r^5t^3x^7 + 2r^4t^4x^8 - s^2t^4x^8 + 2r^3t^5x^9 + r^2t^6x^{10} - 2rtx^2 + 2r^4sx^3 - 2rt^3x^5 - 2st^2x^4 + 2r^5tx^4 + 4rt^5x^8 + 4st^4x^7 + 10r^2st^2x^5 + 8r^3s^2tx^5 - 2rs^2t^3x^7 + 8r^4st^2x^6 + 10r^3st^3x^7 - 2rs^3t^3x^8 + 4r^2st^4x^8 + 2rstx^3 + 10r^2s^2t^2x^6 - 4r^2s^3t^2x^7 - 2r^3s^2t^3x^8 + 6r^3stx^4 + 6rs^3tx^5 + 12rst^3x^6 - 2rs^4tx^6 + 2r^5stx^5 + 4rst^5x^9 + 1)W_1^2,$

$\Gamma_9 = t^2x^3(-2r^2x + 2s^2x^2 + 2s^3x^3 - 6t^2x^3 - s^4x^4 + 3t^4x^6 - 6sx + 2r^2s^2x^3 + 9r^2t^2x^4 + 2r^4t^2x^5 + 6s^2t^2x^5 + 4r^3t^3x^6 + 2r^2t^4x^7 - 4s^3t^2x^6 + s^2t^4x^8 - 6rtx^2 + 2r^2sx^2 + 4r^3tx^3 + 6rt^3x^5 + 6st^2x^4 + 12r^2st^2x^5 - 2rs^2t^3x^7 + 6rstx^3 - r^2s^2t^2x^6 + 6rs^2tx^4 + 4r^3stx^4 - 2rs^3tx^5 + 12rst^3x^6 + 3)W_0^2,$

$\Gamma_{10} = -2x^3(-3rs + r^3s^2x^2 + 6r^3t^2x^3 + 5r^2t^3x^4 + r^5t^2x^4 + 2r^4t^3x^5 + 2r^3t^4x^6 + 2r^2t^5x^7 - s^3t^3x^6 + 2rs^2x + 2r^3sx - 4r^2tx + rs^3x^2 - 5rt^2x^2 + 3r^4tx^2 + 3s^2tx^2 + 2s^3tx^3 + 4rt^4x^5 + 2st^3x^4 - s^4tx^4 + rt^6x^8 + 2st^5x^7 - 4stx + 4r^2s^2tx^3 + 8rs^2t^2x^4 + 7r^3st^2x^4 - 4rs^3t^2x^5 + 10r^2st^3x^5 - 2r^2s^2t^3x^6 + 7r^2stx^2 + 6rst^2x^3 + 2r^4stx^3 + 3rst^4x^6)$
$W_2W_1,$

$\Gamma_{11} = -2tx^3(-3r + 2r^3x - r^3t^2x^4 + 2r^2t^3x^5 - s^2t^3x^6 - 5stx^2 + rs^2x^2 + r^3sx^2 + r^2tx^2 + 4s^2tx^3 + 3s^3tx^4 + 3rt^4x^6 + 4st^3x^5 - 2s^4tx^5 + st^5x^8 + 2rsx + 4r^2s^2tx^4 + 6rs^2t^2x^5 + 2r^3st^2x^5 - rs^3t^2x^6 + r^2st^3x^6 + 4r^2stx^3 + 4rst^2x^4)$
$W_2W_0,$

$\Gamma_{12} = 2tx^3(3s - 6s^2x + 2s^3x^2 + 2s^4x^3 - s^5x^4 + 2r^2s^2x^2 + 2r^2s^3x^3 - 2r^2t^2x^3 - r^4t^2x^4 + s^2t^2x^4 - r^2t^4x^6 + 2s^3t^2x^5 - s^4t^2x^6 + 2s^2t^4x^7 - 2r^2sx - 3r^3tx^2 - 2rt^3x^4 - 2rt^5x^7 - 3st^4x^6 + 4rtx + 4r^3s^2tx^4 + rs^2t^3x^6 + 2r^4st^2x^5 + r^3st^3x^6 - 9rstx^2 + 8r^2s^2t^2x^5 - r^2s^3t^2x^6 + 4rs^2tx^3 + 2r^3stx^3 + 7rs^3tx^4 + 2rst^3x^5 - 2rs^4tx^5 + rst^5x^8)W_1W_0$

and

$\Phi_1 = x^{n+3}(n(r + stx^2)(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 2r^3x - 3r - r^3t^2x^4 + 2r^2t^3x^5 - s^2t^3x^6 - 5stx^2 + rs^2x^2 + r^3sx^2 + r^2tx^2 + 4s^2tx^3 + 3s^3tx^4 + 3rt^4x^6 + 4st^3x^5 - 2s^4tx^5 + st^5x^8 + 2rsx + 4r^2s^2tx^4 + 6rs^2t^2x^5 + 2r^3st^2x^5 - rs^3t^2x^6 + r^2st^3x^6 + 4r^2stx^3 + 4rst^2x^4)W_{n+3}^2,$

$\Phi_2 = x^{n+4}(t + rs)(n(s + rtx)(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 3s^2x - 4s + 2s^3x^2 - s^4x^3 + 2r^2s^2x^2 + 3r^2t^2x^3 + 2r^4t^2x^4 + r^3t^3x^5 - s^3t^2x^5 + 3r^2sx + 4r^3tx^2 + 4rt^3x^4 + 2st^2x^3 + rt^5x^7 + 2st^4x^6 - 5rtx + 8r^2st^2x^4 + 6rstx^2 - r^2s^2t^2x^5 + 7rs^2tx^3 + 4r^3stx^3 - 2rs^3tx^4)W_{n+2}^2,$

$\Phi_3 = t^2x^{n+4}(n(r + stx^2)(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 3r^3x - 4r + r^2t^3x^5 - 2s^2t^3x^6 + s^3t^3x^7 - 6stx^2 + 2rs^2x^2 + 2r^3sx^2 - rs^3x^3 + 2r^2tx^2 + 2rt^2x^3 + r^4tx^3 + 5s^2tx^3 + 4s^3tx^4 + 2rt^4x^6 + 6st^3x^5 - 3s^4tx^5 + 3rsx + 4r^2s^2tx^4 + 11rs^2t^2x^5 + 3r^3st^2x^5 - 2rs^3t^2x^6 + 2r^2st^3x^6 + 9r^2stx^3 + 4rst^2x^4 - rst^4x^7)W_{n+1}^2,$

$\Phi_4 = x^{n+2}(n(t^2x^3 - sx - rtx^2 - 1)(r^2x + s^2x^2 + t^2x^3 + 2rstx^3 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 4r^2x - 2r^4x^2 + 4s^2x^2 - 2s^3x^3 + 3t^2x^3 - 2s^4x^4 + s^5x^5 - t^6x^9 + sx - 4r^2s^2x^3 - 2r^2s^3x^4 - 6r^2t^2x^4 + r^4t^2x^5 - 8s^2t^2x^5 - 4r^3t^3x^6 - 4r^2t^4x^7 + 2s^3t^2x^6 + s^4t^2x^7 - 2s^2t^4x^8 - 2r^2sx^2 - r^4sx^3 - 2r^3tx^3 - 2st^2x^4 + st^4x^7 - 10r^2st^2x^5 - 8r^3s^2tx^5 + 2rs^2t^3x^7 - 4r^4st^2x^6 - 2r^3st^3x^7 + 6rstx^3 - 14r^2s^2t^2x^6 + 2r^2s^3t^2x^7 - 8rs^2tx^4 - 8r^3stx^4 - 10rs^3tx^5 - 16rst^3x^6 + 4rs^4tx^6 - 2rst^5x^9 - 2)W_{n+3}W_{n+2},$

$\Phi_5 = tx^{n+3}(n(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 - t^2x^3 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 3r^4x^2 - 2r^2x + 6s^2x^2 - 4s^3x^3 + 6t^2x^3 - 3s^4x^4 + 2s^5x^5 - 3t^4x^6 + 2sx - 2r^2s^2x^3 - 4r^2s^3x^4 - 4r^2t^2x^4 - 10s^2t^2x^5 - 2r^3t^3x^6 + 4s^3t^2x^6 - 2s^2t^4x^8 + rtx^2 - 2rt^3x^5 - 4st^2x^4 + r^5tx^4 + rt^5x^8 + 2st^4x^7 - 4r^2st^2x^5 - 2r^3s^2tx^5 + 2rs^2t^3x^7 - 2r^2s^2t^2x^6 - 2rs^2tx^4 + 4r^3stx^4 - 8rs^3tx^5 - 12rst^3x^6 + rs^4tx^6 - 3 + 4r^2sx^2 + 2r^4sx^3 + 2r^3tx^3)W_{n+3}W_{n+1},$

$\Phi_6 = x^{n+1}(n(t^2x^3 - sx - rtx^2 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + sx + rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + 2rstx^3 - rs^2tx^4 - 1) + 2r^2x - r^4x^2 + s^2x^2 - 4s^3x^3 + s^4x^4 + 2s^5x^5 - s^6x^6 + 3t^4x^6 - 2t^6x^9 + 2sx - 4r^2s^2x^3 - 5r^2t^2x^4 - r^4s^2x^4 + 2r^2s^4x^5 - 4s^2t^2x^5 - 2r^2t^4x^7 + 8s^3t^2x^6 - r^6t^2x^6 + 2r^4t^4x^8 - 3s^2t^4x^8 - s^5t^2x^8 - r^2t^6x^{10} + 2s^3t^4x^9 + 2rtx^2 - 3st^2x^4 - 2r^5tx^4 + 2rt^5x^8 + st^6x^{10} - 2r^4sx^3 - 4rt^3x^5 - 2r^2st^2x^5 - 4r^3s^2tx^5 + 16rs^2t^3x^7 + 4r^3s^3tx^6 - 5r^4st^2x^6 + 4r^3st^3x^7 - 2rs^3t^3x^8 + 8r^2st^4x^8 + 2rs^2t^5x^{10} - 2rstx^3 + 10r^2s^3t^2x^7 + 2r^4s^2t^2x^7 - r^2s^4t^2x^8 + 2r^3s^2t^3x^8 - 2r^2s^2t^4x^9 - 6rs^2tx^4 - 6r^3stx^4 + 8rs^4tx^6 - 2r^5stx^5 - 2rs^5tx^7 - 6rst^5x^9 - 1 - 4rst^3x^6)W_{n+2}W_{n+1},$

$\Phi_7 = x^2(2r - r^3x + 2r^3t^2x^4 - 3r^2t^3x^5 + s^3t^3x^7 + 4stx^2 - rs^3x^3 + 2rt^2x^3 + r^4tx^3 - 3s^2tx^3 - 2s^3tx^4 - 4rt^4x^6 - 2st^3x^5 + s^4tx^5 - 2st^5x^8 - rsx - 4r^2s^2tx^4 - rs^2t^2x^5 - r^3st^2x^5 + r^2stx^3 - 4rst^2x^4 - rst^4x^7)W_2^2,$

$\Phi_8 = x^3(t + rs)(3s - 2s^2x - s^3x^2 - r^2s^2x^2 - 2r^2t^2x^3 - r^4t^2x^4 - s^2t^2x^4 - r^2t^4x^6 + 2s^3t^2x^5 - 2r^2sx - 3r^3tx^2 - 2rt^3x^4 - 2rt^5x^7 - 3st^4x^6 + 4rtx - 3r^2st^2x^4 + rs^2t^3x^6 - 4rstx^2 - 2rs^2tx^3 - 2r^3stx^3 - 2rst^3x^5)W_1^2,$

$\Phi_9 = t^2x^3(3r - 2r^3x + r^3t^2x^4 - 2r^2t^3x^5 + s^2t^3x^6 + 5stx^2 - rs^2x^2 - r^3sx^2 - r^2tx^2 - 4s^2tx^3 - 3s^3tx^4 - 3rt^4x^6 - 4st^3x^5 + 2s^4tx^5 - st^5x^8 - 2rsx - 4r^2s^2tx^4 - 6rs^2t^2x^5 - 2r^3st^2x^5 + rs^3t^2x^6 - r^2st^3x^6 - 4r^2stx^3 - 4rst^2x^4) W_0^2,$

$\Phi_{10} = -x(2r^2x - r^4x^2 + 2s^2x^2 - s^4x^4 + 3t^4x^6 - 2t^6x^9 - 2r^2s^2x^3 - 2r^2s^3x^4 - 4r^2t^2x^4 + 2r^4t^2x^5 - 6s^2t^2x^5 - 4r^3t^3x^6 - 4r^2t^4x^7 + 2s^4t^2x^7 - 2s^2t^4x^8 - rtx^2 - 2r^2sx^2 - 2r^3tx^3 + 2rt^3x^5 + r^5tx^4 - rt^5x^8 - 8r^2st^2x^5 - 6r^3s^2tx^5 - 2rs^2t^3x^7 - 2r^4st^2x^6 + 2rs^3t^3x^8 - 2r^2st^4x^8 - 4r^2s^2t^2x^6 - 4rs^2tx^4 - 2r^3stx^4 - 4rs^3tx^5 - 8rst^3x^6 + rs^4tx^6 - 4rst^5x^9 - 1)W_2W_1,$

$\Phi_{11} = tx^2(2r^2x - 2r^4x^2 - 4s^2x^2 + 2s^3x^3 - 3t^2x^3 + 2s^4x^4 - s^5x^5 + t^6x^9 - sx + 2r^2s^2x^3 + 2r^2s^3x^4 + 6r^2t^2x^4 + r^4t^2x^5 + 8s^2t^2x^5 - 2r^2t^4x^7 - 2s^3t^2x^6 - s^4t^2x^7 + 2s^2t^4x^8 - 2r^2sx^2 - r^4sx^3 + 2st^2x^4 - st^4x^7 + 2r^2st^2x^5 + 4rstx^3 + 2r^2s^2t^2x^6 + 4rs^3tx^5 + 8rst^3x^6 + 2)W_2W_0,$

$\Phi_{12} = tx^3(3t - 6t^3x^3 + 3t^5x^6 + 6rs - 2r^3s^2x^2 - 2r^3t^2x^3 + 4r^2t^3x^4 - r^5t^2x^4 + 2s^2t^3x^5 + 2r^3t^4x^6 - 2s^3t^3x^6 - 4rs^2x - 4r^3sx + 2r^2tx - 2rs^3x^2 - rt^2x^2 - 3r^4tx^2 + 4s^2tx^2 - 4s^3tx^3 + 2rt^4x^5 + 4st^3x^4 - 3s^4tx^4 + 2s^5tx^5 - rt^6x^8 - 2st^5x^7 - 2stx - 6r^2s^2tx^3 - 6rs^2t^2x^4 - 4r^2s^3tx^4 - 2r^3st^2x^4 - 4rs^3t^2x^5 + rs^4t^2x^6 - 2rs^2t^4x^7 - 2r^3s^2t^2x^5 - 6r^2stx^2 - 2r^4stx^3 + 6rst^4x^6)W_1W_0$

and

$\Psi_1 = (n(-t^2x^3 + sx + rtx^2 + 1)(s - s^2x + r^2 + rtx)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + 2r^4x - 3s + 6s^2x - 2s^3x^2 - 2s^4x^3 + s^5x^4 - 3r^2 - r^2s^2x^2 - 2r^2s^3x^3 + 2r^2t^2x^3 - s^2t^2x^4 + 2r^3t^3x^5 + 4r^2t^4x^6 - 2s^3t^2x^5 + s^4t^2x^6 - 2s^2t^4x^7 + 4r^2sx + r^4sx^2 + 4r^3tx^2 + 2rt^3x^4 + 2rt^5x^7 + 3st^4x^6 - 4rtx + 4r^2st^2x^4 - 2rs^2t^3x^6 + 4rstx^2 - 2r^2s^2t^2x^5 + 2r^3stx^3 - 4rs^3tx^4 + 2rst^3x^5)x^{n+3}W_{n+3}^2,$

$\Psi_2 = (n(t^2x^3 - sx - rtx^2 - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)(-s + s^2x - r^2t^2x^3 + r^2sx - rt^3x^4 + rs^2tx^3) + 4s^2x - 2s - 2s^3x^2 - 4r^2s^2x^2 - 2r^2s^3x^3 - 5r^2t^2x^3 - r^4s^2x^3 + 4r^4t^2x^4 + 2s^2t^2x^4 + 8r^3t^3x^5 + 8r^2t^4x^6 - 4s^3t^2x^5 + 2r^5t^3x^6 + 2s^4t^2x^6 + 4r^4t^4x^7 - 3s^2t^4x^7 + 2r^3t^5x^8 - 2r^4sx^2 - 6rt^3x^4 - 2st^2x^3 + 6rt^5x^7 + 4st^4x^6 + 2r^2st^2x^4 - 4r^3s^2tx^4 - 2rs^2t^3x^6 - 4r^3s^3tx^5 + 4r^4st^2x^5 + 10r^3st^3x^6 - 2rs^3t^3x^7 + 8r^2st^4x^7 - 8r^2s^3t^2x^6 - 2r^4s^2t^2x^6 + r^2s^4t^2x^7 - 2r^3s^2t^3x^7 - 2r^2s^2t^4x^8 - 2r^3stx^3 - 2rs^3tx^4 + 8rst^3x^5 - 4rs^4tx^5 + 2rs^5tx^6 - 2rst^5x^8 + 4r^2sx)x^{n+2}W_{n+2}^2,$

$\Psi_3 = (n(-t^2x^3 + sx + rtx^2 + 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)(s - s^2x + r^2 + rtx) + 3r^4x - 4s + 8s^2x - 2s^3x^2 - 4s^4x^3 + 2s^5x^4 - 4r^2 - 4r^2s^3x^3 + 5r^2t^2x^3 + 2r^4t^2x^4 - 4s^2t^2x^4 + 2r^3t^3x^5 + 2r^2t^4x^6 - s^2$

$t^4x^7+6r^2sx+2r^4sx^2+6r^3tx^2+4rt^3x^4+2st^2x^3+r^5tx^3+rt^5x^7+2st^4x^6-5rtx+8r^2st^2x^4-2r^3s^2tx^4+6rstx^2-3r^2s^2t^2x^5+4rs^2tx^3+8r^3stx^3-10rs^3tx^4+rs^4tx^5)t^2x^{n+4}W_{n+1}^2,$

$$\Psi_4=(n(r+tx)(t^2x^3-sx-rtx^2-1)(r^2x-s^2x^2+t^2x^3-1)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)+4r^3x-2r-2r^5x^2+6t^3x^4-3t^5x^7-3tx+2r^3s^2x^3+2r^3s^3x^4-8r^3t^2x^4-8r^2t^3x^5-4r^4t^3x^6-2s^2t^3x^6-8r^3t^4x^7-4r^2t^5x^8+2s^3t^3x^7+2stx^2-4rs^2x^2-2r^3sx^2+4rs^3x^3+6r^2tx^2+4rt^2x^3+2rs^4x^4-r^5sx^3-5r^4tx^3-4s^2tx^3-rs^5x^5+4s^3tx^4-2rt^4x^6-4st^3x^5+3s^4tx^5-2s^5tx^6+2st^5x^8+rsx+6r^2s^2tx^4+8r^2s^3tx^5-8r^3st^2x^5+10rs^3t^2x^6-12r^2st^3x^6-2rs^4t^2x^7+4rs^2t^4x^8+4r^3s^2t^2x^6+4r^2s^2t^3x^7-6r^2stx^3-2rst^2x^4-2r^4stx^4-11rst^4x^7)x^{n+2}W_{n+3}W_{n+2},$$

$$\Psi_5=(n(t^2x^3-sx-rtx^2-1)(r^2x+s^2x^2-s^3x^3+t^2x^3+sx+r^2sx^2-st^2x^4+2rstx^3-1)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)+2r^2x-r^4x^2+s^2x^2-4s^3x^3+s^4x^4+2s^5x^5-s^6x^6+3t^4x^6-2t^6x^9+2sx-4r^2s^2x^3-4r^2t^2x^4-r^4s^2x^4+2r^2s^4x^5+2r^4t^2x^5-4s^2t^2x^5-4r^3t^3x^6-4r^2t^4x^7+8s^3t^2x^6-3s^2t^4x^8-s^5t^2x^8+2s^3t^4x^9-rtx^2-2r^4sx^3-2r^3tx^3+2rt^3x^5-3st^2x^4+r^5tx^4-rt^5x^8+st^6x^{10}-2r^2st^2x^5-6r^3s^2tx^5+6rs^2t^3x^7-r^4st^2x^6+2rs^3t^3x^8-4r^2st^4x^8-2r^2s^2t^2x^6+2r^2s^3t^2x^7-2r^3stx^4-4rs^3tx^5-8rst^3x^6+5rs^4tx^6-4rst^5x^9-1)x^{n+1}W_{n+3}W_{n+1},$$

$$\Psi_6=(n(sx-1)(-t^2x^3+sx+rtx^2+1)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)(r^2x-s^2x^2+t^2x^3-1)+4r^2x-2r^4x^2-6s^2x^2+8s^3x^3+3t^2x^3-2s^4x^4-4s^5x^5+2s^6x^6-t^6x^9+4sx+4r^2s^2x^3-6r^2t^2x^4+2r^4s^2x^4-4r^2s^4x^5+r^4t^2x^5-4r^3t^3x^6-4r^2t^4x^7+4s^3t^2x^6-3s^4t^2x^7-8r^2sx^2+2r^4sx^3-2r^3tx^3-8st^2x^4+4st^4x^7+4r^2st^2x^5+4r^3s^2tx^5+14rs^2t^3x^7-2r^3s^3tx^6+4r^3st^3x^7-2rs^3t^3x^8+4r^2st^4x^8-5rstx^3+6r^2s^2t^2x^6-2r^2s^3t^2x^7+4rs^2tx^4+2r^3stx^4-6rst^3x^6-8rs^4tx^6+r^5stx^5+rs^5tx^7-rst^5x^9-2)tx^{n+2}W_{n+2}W_{n+1},$$

$$\Psi_7=(2s-r^4x-4s^2x+2s^3x^2+2r^2+2r^2s^2x^2+r^2t^2x^3+2r^4t^2x^4-2s^2t^2x^4-2r^3t^3x^5-6r^2t^4x^6+4s^3t^2x^5-2s^4t^2x^6+3s^2t^4x^7-2r^2sx-2r^3tx^2+2st^2x^3+r^5tx^3-3rt^5x^7-4st^4x^6+3rtx-2r^3s^2tx^4+4rs^2t^3x^6-2rstx^2+r^2s^2t^2x^5+4rs^2tx^3+4r^3stx^3-2rs^3tx^4-4rst^3x^5+rs^4tx^5)x^2W_2^2$$

$$\Psi_8=x(s-2s^2x+2s^3x^2-2s^4x^3+s^5x^4+3r^2s^2x^2+4r^2t^2x^3+r^2s^4x^4-3r^4t^2x^4-5s^2t^2x^4-6r^3t^3x^5-5r^2t^4x^6+6s^3t^2x^5-r^5t^3x^6-3s^4t^2x^6-2r^4t^4x^7+4s^2t^4x^7-2r^3t^5x^8-2r^2t^6x^9-2r^2sx+r^4sx^2+5rt^3x^4+4st^2x^3-4rt^5x^7-5st^4x^6-rt^7x^{10}-2r^2st^2x^4-2r^3s^2tx^4+2rs^2t^3x^6+4r^3s^3tx^5-4r^4st^2x^5-4r^3st^3x^6+2rs^3t^3x^7-4r^2st^4x^7-rs^4t^3x^8+2rs^2t^5x^9+rstx^2+2r^2s^3t^2x^6+r^4s^2t^2x^6+3r^2s^2t^4x^8+4rs^2tx^3+2r^3stx^3-4rs^3tx^4-8rst^3x^5+4rs^4tx^5-r^5stx^4-rs^5tx^6+rst^5x^8)W_1^2,$$

$$\Psi_9=t^2x^3(3s-2r^4x-6s^2x+2s^3x^2+2s^4x^3-s^5x^4+3r^2+r^2s^2x^2+2r^2s^3x^3-2r^2t^2x^3+s^2t^2x^4-2r^3t^3x^5-4r^2t^4x^6+2s^3t^2x^5-s^4t^2x^6+2s^2t^4x^7-4r^2sx-r^4sx^2-4r^3tx^2-2rt^3x^4-2rt^5x^7-3st^4x^6+4rtx-4r^2st^2x^4+2rs^2t^3x^6-4rstx^2+2r^2s^2t^2x^5-2r^3stx^3+4rs^3tx^4-2rst^3x^5)W_0^2,$$

$$\Psi_{10}=x(-r^6tx^4-2r^5t^2x^5+r^5x^2+2r^4s^2tx^5-3r^4stx^4+3r^4t^3x^6+4r^4tx^3-2r^3s^2t^2x^6-2r^3s^2x^3+4r^3st^2x^5+2r^3sx^2+8r^3t^4x^7+6r^3t^2x^4-2r^3x-r^2s^4tx^6-2r^2s^3tx^5-4r^2s^2t^3x^7-6r^2s^2tx^4+8r^2st^3x^6+10r^2stx^3+5r^2t^5x^8+4r^2t^3x^5-3r^2tx^2+2rs^4t^2x^7-rs^4x^4-6rs^3t^2x^6-4rs^3x^3-6rs^2t^4x^8+2rs^2t^2x^5+4rs^2x^2+8rst^4x^7+4rst^2x^4+2rt^6x^9-3rt^4x^6+r+s^5tx^6+s^4t^3x^8-2s^4tx^5-2s^3t^3x^7-4s^3tx^4-2s^2t^5x^9+4s^2t^3x^6+4s^2tx^3-st^5x^8+2st^3x^5-stx^2+t^7x^{10}-3t^3x^4+2tx)W_2W_1,$$

$$\Psi_{11}=tx^2(2r+2r^3x-2r^5x^2-6t^3x^4+3t^5x^7+3tx+2r^3s^3x^4+4r^3t^2x^4+4r^2t^3x^5+2s^2t^3x^6-2s^3t^3x^7-2stx^2-8rs^2x^2-6r^3sx^2+2r^2tx^2-4rt^2x^3+2rs^4x^4-r^5sx^3-3r^4tx^3+4s^2tx^3-rs^5x^5-4s^3tx^4+2rt^4x^6+4st^3x^5-3s^4tx^5+2s^5tx^6-2st^5x^8+5rsx-6r^2s^2tx^4+2rs^2t^2x^5-6rs^3t^2x^6+8r^2st^3x^6-2r^2stx^3+2rst^2x^4-2r^4stx^4+5rst^4x^7)W_2W_0,$$

$\Psi_{12} = tx(-r^5tx^4 - r^4s^2x^4 + r^4st^2x^6 - 2r^4sx^3 - 2r^4t^2x^5 + r^4x^2 + 2r^3s^2tx^5 - 4r^3st^3x^7 - 6r^3stx^4 + 4r^3t^3x^6 + 2r^3tx^3 + 2r^2s^4x^5 + 2r^2s^3t^2x^7 + 2r^2s^3x^4 - 6r^2s^2t^2x^6 - 4r^2s^2x^3 - 4r^2st^4x^8 - 2r^2st^2x^5 + 6r^2sx^2 + 4r^2t^4x^7 + 4r^2t^2x^4 - 2r^2x + 3rs^4tx^6 + 2rs^3t^3x^8 + 4rs^3tx^5 - 10rs^2t^3x^7 - 8rs^2tx^4 + 4rst^3x^6 + 8rstx^3 + rt^5x^8 - 2rt^3x^5 + rtx^2 - s^6x^6 - s^5t^2x^8 + 2s^5x^5 + 4s^4t^2x^7 + 3s^4x^4 + 2s^3t^4x^9 - 6s^3t^2x^6 - 8s^3x^3 - 3s^2t^4x^8 + 4s^2t^2x^5 + 5s^2x^2 - st^6x^{10} + 3st^2x^4 - 2sx + 2t^6x^9 - 3t^4x^6 + 1)W_1W_0.$

Proof. First, we obtain $\sum_{k=0}^{n} kx^kW_k^2$. Using the recurrence relation

$$W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n$$

or

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

i.e.

$$t^2W_n^2 = (W_{n+3} - rW_{n+2} - sW_{n+1})^2 = W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 - 2rW_{n+3}W_{n+2} - 2sW_{n+3}W_{n+1} + 2rsW_{n+2}W_{n+1}$$

we obtain

$$\begin{aligned}
t^2 \times n \times x^nW_n^2 &= n \times x^nW_{n+3}^2 + r^2 \times n \times x^nW_{n+2}^2 + s^2 \times n \times x^nW_{n+1}^2 \\
&\quad -2r \times n \times x^nW_{n+3}W_{n+2} - 2s \times n \times x^nW_{n+3}W_{n+1} + 2rs \times n \times x^nW_{n+2}W_{n+1} \\
t^2 \times (n-1) \times x^{n-1}W_{n-1}^2 &= (n-1) \times x^{n-1}W_{n+2}^2 + r^2 \times (n-1) \times x^{n-1}W_{n+1}^2 + s^2 \times (n-1) \times x^{n-1}W_n^2 \\
&\quad -2r \times (n-1) \times x^{n-1}W_{n+2}W_{n+1} - 2s \times (n-1) \times x^{n-1}W_{n+2}W_n \\
&\quad +2rs \times (n-1) \times x^{n-1}W_{n+1}W_n \\
t^2 \times (n-2) \times x^{n-2}W_{n-2}^2 &= (n-2) \times x^{n-2}W_{n+1}^2 + r^2 \times (n-2)x^{n-2}W_n^2 + s^2 \times (n-2) \times x^{n-2}W_{n-1}^2 \\
&\quad -2r \times (n-2) \times x^{n-2}W_{n+1}W_n - 2s \times (n-2) \times x^{n-2}W_{n+1}W_{n-1} \\
&\quad +2rs \times (n-2)x^{n-2}W_nW_{n-1} \\
&\quad \vdots \\
t^2 \times 1 \times x^1W_1^2 &= 1 \times x^1W_4^2 + r^2 \times 1 \times x^1W_3^2 + s^2 \times 1 \times x^1W_2^2 \\
&\quad -2r \times 1 \times x^1W_4W_3 - 2s \times 1 \times x^1W_4W_2 + 2rs \times 1 \times x^1W_3W_2 \\
t^2 \times 0 \times x^0W_0^2 &= 0 \times x^0W_3^2 + r^2 \times 0 \times x^0W_2^2 + s^2 \times 0 \times x^0W_1^2 \\
&\quad -2r \times 0 \times x^0W_3W_2 - 2s \times 0 \times x^0W_3W_1 + 2rs \times 0 \times x^0W_2W_1
\end{aligned}$$

If we add the equations side by side, we get

$$
\begin{aligned}
t^2\sum_{k=0}^{n}kx^kW_k^2 &= (nx^nW_{n+3}^2+(n-1)x^{n-1}W_{n+2}^2+(n-2)x^{n-2}W_{n+1}^2 \\
&+1\times x^{-1}W_2^2+2\times x^{-2}W_1^2+3\times x^{-3}W_0^2+\sum_{k=0}^{n}(k-3)x^{k-3}W_k^2) \\
&+r^2(nx^nW_{n+2}^2+(n-1)x^{n-1}W_{n+1}^2+1\times x^{-1}W_1^2+2\times x^{-2}W_0^2+\sum_{k=0}^{n}(k-2)x^{k-2}W_k^2) \\
&+s^2(nx^nW_{n+1}^2+1\times x^{-1}W_0^2+\sum_{k=0}^{n}(k-1)x^{k-1}W_k^2)-2r(nx^nW_{n+3}W_{n+2} \\
&+(n-1)x^{n-1}W_{n+2}W_{n+1}+1\times x^{-1}W_2W_1+2\times x^{-2}W_1W_0+\sum_{k=0}^{n}(k-2)x^{k-2}W_{k+1}W_k) \\
&-2s(nx^nW_{n+3}W_{n+1}+1\times x^{-1}W_2W_0+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k+2}W_k) \\
&+2rs(nx^nW_{n+2}W_{n+1}+1\times x^{-1}W_1W_0+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k+1}W_k)
\end{aligned}
$$

and so

$$
\begin{aligned}
t^2\sum_{k=0}^{n}kx^kW_k^2 &= x^{-3}(r^2x+s^2x^2+1)\sum_{k=0}^{n}kx^kW_k^2-x^{-3}(2r^2x+s^2x^2+3)\sum_{k=0}^{n}x^kW_k^2 \qquad (2.1) \\
&-2sx^{-1}\sum_{k=0}^{n}kx^kW_kW_{k+2}+2sx^{-1}\sum_{k=0}^{n}x^kW_kW_{k+2}+2r(sx-1)x^{-2}\sum_{k=0}^{n}kx^kW_{k+1}W_k \\
&+2r(2-sx)x^{-2}\sum_{k=0}^{n}x^kW_{k+1}W_k+nx^nW_{n+3}^2+(n+nr^2x-1)x^{n-1}W_{n+2}^2 \\
&+(n-r^2x+nr^2x+ns^2x^2-2)x^{n-2}W_{n+1}^2-2nrx^nW_{n+2}W_{n+3} \\
&-2nsx^nW_{n+1}W_{n+3}+2r(-n+nsx+1)x^{n-1}W_{n+1}W_{n+2} \\
&+x^{-1}W_2^2+(r^2x+2)x^{-2}W_1^2+(2r^2x+s^2x^2+3)x^{-3}W_0^2 \\
&-2rx^{-1}W_1W_2-2sx^{-1}W_0W_2+2r(sx-2)x^{-2}W_1W_0
\end{aligned}
$$

Next we obtain $\sum_{k=0}^{n}kx^kW_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$tW_n=W_{n+3}-rW_{n+2}-sW_{n+1}$$

by W_{n+1} we get

$$tW_{n+1}W_n=W_{n+3}W_{n+1}-rW_{n+2}W_{n+1}-sW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{array}{rcl}
t\times n\times x^{n}W_{n+1}W_{n} & = & nx^{n}W_{n+3}W_{n+1}-r\times n\times x^{n}W_{n+2}W_{n+1}-s\times n\times x^{n}W_{n+1}^{2}\\
t\times (n-1)\times x^{n-1}W_{n}W_{n-1} & = & (n-1)\times x^{n-1}W_{n+2}W_{n}-r\times (n-1)\times x^{n-1}W_{n+1}W_{n}\\
 & & -s\times (n-1)\times x^{n-1}W_{n}^{2}\\
t\times (n-2)\times x^{n-2}W_{n-1}W_{n-2} & = & (n-2)\times x^{n-2}W_{n+1}W_{n-1}-r\times (n-2)\times x^{n-2}W_{n}W_{n-1}\\
 & & -s\times (n-2)\times x^{n-2}W_{n-1}^{2}\\
 & & \vdots\\
t\times 2\times x^{2}W_{3}W_{2} & = & 2\times x^{2}W_{5}W_{3}-r\times 2\times W_{4}W_{3}-s\times 2\times x^{2}W_{3}^{2}\\
t\times 1\times xW_{2}W_{1} & = & 1\times xW_{4}W_{2}-r\times 1\times xW_{3}W_{2}-s\times 1\times xW_{2}^{2}\\
t\times 0\times x^{0}W_{1}W_{0} & = & 0\times x^{0}W_{3}W_{1}-r\times 0\times x^{0}W_{2}W_{1}-s\times 0\times x^{0}W_{1}^{2}
\end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl}
t\sum_{k=0}^{n}kx^{k}W_{k+1}W_{k} & = & (nx^{n}W_{n+3}W_{n+1}+1\times x^{-1}W_{2}W_{0}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k+2}W_{k})\\
 & & -r(nx^{n}W_{n+2}W_{n+1}+1\times x^{-1}W_{1}W_{0}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k+1}W_{k})\\
 & & -s(nx^{n}W_{n+1}^{2}+1\times x^{-1}W_{0}^{2}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k}^{2})
\end{array}$$

and so

$$\begin{array}{rcl}
t\sum_{k=0}^{n}kx^{k}W_{k+1}W_{k} & = & -sx^{-1}\sum_{k=0}^{n}kx^{k}W_{k}^{2}+sx^{-1}\sum_{k=0}^{n}x^{k}W_{k}^{2}+x^{-1}\sum_{k=0}^{n}kx^{k}W_{k}W_{k+2}\\
 & & -x^{-1}\sum_{k=0}^{n}x^{k}W_{k}W_{k+2}-rx^{-1}\sum_{k=0}^{n}kx^{k}W_{k}W_{k+1}+rx^{-1}\sum_{k=0}^{n}x^{k}W_{k}W_{k+1}\\
 & & -nsx^{n}W_{n+1}^{2}+nx^{n}W_{n+3}W_{n+1}-nrx^{n}W_{n+2}W_{n+1}-\frac{s}{x}W_{0}^{2}+\frac{1}{x}W_{2}W_{0}-\frac{r}{x}W_{1}W_{0}
\end{array} \tag{2.2}$$

Next we obtain $\sum_{k=0}^{n}kx^{k}W_{k+2}W_{k}$. Multiplying the both side of the recurrence relation

$$tW_{n}=W_{n+3}-rW_{n+2}-sW_{n+1}$$

by W_{n+2} we get

$$tW_{n+2}W_{n}=W_{n+3}W_{n+2}-rW_{n+2}^{2}-sW_{n+2}W_{n+1}.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
t\times n\times x^{n}W_{n+2}W_{n} &= nx^{n}W_{n+3}W_{n+2}-r\times n\times x^{n}W_{n+2}^{2}-s\times n\times x^{n}W_{n+2}W_{n+1}\\
t\times (n-1)\times x^{n-1}W_{n+1}W_{n-1} &= (n-1)\times x^{n-1}W_{n+2}W_{n+1}-r\times (n-1)\times x^{n-1}W_{n+1}^{2}\\
&\quad -s\times (n-1)\times x^{n-1}W_{n+1}W_{n}\\
t\times (n-2)x^{n-2}W_{n}W_{n-2} &= (n-2)\times x^{n-2}W_{n+1}W_{n}-r\times (n-2)\times x^{n-2}W_{n}^{2}-s\times (n-2)\times x^{n-2}W_{n}W_{n-1}\\
&\ \ \vdots\\
t\times 2\times x^{2}W_{4}W_{2} &= 2\times x^{2}W_{5}W_{4}-r\times 2\times x^{2}W_{4}^{2}-s\times 2\times x^{2}W_{4}W_{3}\\
t\times 1\times x^{1}W_{3}W_{1} &= 1\times x^{1}W_{4}W_{3}-r\times 1\times x^{1}W_{3}^{2}-s\times 1\times x^{1}W_{3}W_{2}\\
t\times 0\times x^{0}W_{2}W_{0} &= 0\times x^{0}W_{3}W_{2}-r\times 0\times x^{0}W_{2}^{2}-s\times 0\times x^{0}W_{2}W_{1}
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
t\sum_{k=0}^{n}kx^{k}W_{k+2}W_{k} &= (nx^{n}W_{n+3}W_{n+2}+(n-1)x^{n-1}W_{n+2}W_{n+1}+1\times x^{-1}W_{2}W_{1}\\
&\quad +2\times x^{-2}W_{1}W_{0}+\sum_{k=0}^{n}(k-2)x^{k-2}W_{k+1}W_{k})-r(nx^{n}W_{n+2}^{2}+(n-1)x^{n-1}W_{n+1}^{2}\\
&\quad +1\times x^{-1}W_{1}^{2}+2\times x^{-2}W_{0}^{2}+\sum_{k=0}^{n}(k-2)x^{k-2}W_{k}^{2})-s(nx^{n}W_{n+2}W_{n+1}\\
&\quad +1\times x^{-1}W_{1}W_{0}+\sum_{k=0}^{n}(k-1)x^{k-1}W_{k+1}W_{k})
\end{aligned}$$

and so

$$\begin{aligned}
t\sum_{k=0}^{n}kx^{k}W_{k+2}W_{k} &= -rx^{-2}\sum_{k=0}^{n}kx^{k}W_{k}^{2}+2rx^{-2}\sum_{k=0}^{n}x^{k}W_{k}^{2}-(sx-1)x^{-2}\sum_{k=0}^{n}kx^{k}W_{k+1}W_{k} \qquad (2.3)\\
&\quad +(sx-2)x^{-2}\sum_{k=0}^{n}x^{k}W_{k+1}W_{k}-nrx^{n}W_{n+2}^{2}-r(n-1)x^{n-1}W_{n+1}^{2}\\
&\quad +nx^{n}W_{n+3}W_{n+2}+(n-nsx-1)x^{n-1}W_{n+2}W_{n+1}-\frac{r}{x}W_{1}^{2}\\
&\quad -2\frac{r}{x^{2}}W_{0}^{2}+\frac{1}{x}W_{2}W_{1}-(sx-2)x^{-2}W_{1}W_{0}
\end{aligned}$$

Using Theorem 1.1 and solving the system (2.1)-(2.2)-(2.3), the results in (a), (b) and (c) follow.

3. Specific Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}kx^{k}W_{k}^{2}$, $\sum_{k=0}^{n}kx^{k}W_{k+2}W_{k}$ and $\sum_{k=0}^{n}kx^{k}W_{k+1}W_{k}$ for the specific case of sequence $\{W_{n}\}$.

3.1. The Case $x=1$. The case $x=1$ of Theorem 2.1 is given in [30]. In this subsection, we only consider the case $x=1,\ r=0,s=2,t=1$ and $x=1,\ r=1,s=1,t=2$ (these two special cases were not given in [30] because we can not use Theorem 2.1 directly). Observe that setting $x=1,\ r=0,s=2,t=1$ and $x=1,\ r=1,s=1,t=2$ (i.e. for the generalized Pell-Padovan case and for the generalized third order Jacobsthal case) in Theorem 2.1 (a), (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=1,\ r=0,s=2,t=1$ then we have the following theorem (in fact taking $x=1$, $r=0, s=2, t=1$ in Theorem 2.1 and then using L'Hospital rule twice for $x=1$ we obtain the following theorem).

THEOREM 3.1. *If $r=0, s=2, t=1$ then for $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} kW_k^2 = \frac{1}{4}((2n^2+18n+69)W_{n+3}^2+(2n^2+14n+53)W_{n+2}^2+(2n^2+18n+85)W_{n+1}^2-4(n^2+8n+31)W_{n+3}W_{n+2}-4(n^2+10n+40)W_{n+3}W_{n+1}+4(n^2+10n+38)W_{n+2}W_{n+1}-53W_2^2-41W_1^2-69W_0^2-116W_1W_0+124W_2W_0+96W_2W_1)$.

(b): $\sum_{k=0}^{n} kW_{k+1}W_k = \frac{1}{4}(-2(n^2+8n+31)W_{n+3}^2-2(n^2+6n+24)W_{n+2}^2-2(n^2+10n+40)W_{n+1}^2+(4n^2+30n+111)W_{n+3}W_{n+2}+(4n^2+38n+145)W_{n+3}W_{n+1}-(4n^2+38n+137)W_{n+2}W_{n+1}+48W_2^2+38W_1^2+62W_0^2-85W_2W_1-111W_2W_0+103W_1W_0)$.

(c): $\sum_{k=0}^{n} kW_{k+2}W_k = \frac{1}{4}(2(n^2+8n+29)W_{n+3}^2+2(n^2+6n+22)W_{n+2}^2+2(n^2+10n+38)W_{n+1}^2-(4n^2+26n+103)W_{n+3}W_{n+2}-(4n^2+38n+137)W_{n+3}W_{n+1}+(4n^2+34n+125)W_{n+2}W_{n+1}-44W_2^2-34W_1^2-58W_0^2+81W_2W_1+103W_2W_0-95W_1W_0)$.

From Theorem 3.1, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n=R_n$ with $Q_0=1, R_1=1, R_2=1$).

COROLLARY 3.2. *For $n\geq 0$, Pell-Padovan numbers have the following properties:*

(a): $\sum_{k=0}^{n} kR_k^2 = \frac{1}{4}((2n^2+18n+69)R_{n+3}^2+(2n^2+14n+53)R_{n+2}^2+(2n^2+18n+85)R_{n+1}^2-4(n^2+8n+31)R_{n+3}R_{n+2}-4(n^2+10n+40)R_{n+3}R_{n+1}+4(n^2+10n+38)R_{n+2}R_{n+1}-59)$.

(b): $\sum_{k=0}^{n} kR_{k+1}R_k = \frac{1}{4}(-2(n^2+8n+31)R_{n+3}^2-2(n^2+6n+24)R_{n+2}^2-2(n^2+10n+40)R_{n+1}^2+(4n^2+30n+111)R_{n+3}R_{n+2}+(4n^2+38n+145)R_{n+3}R_{n+1}-(4n^2+38n+137)R_{n+2}R_{n+1}+55)$.

(c): $\sum_{k=0}^{n} kR_{k+2}R_k = \frac{1}{4}(2(n^2+8n+29)R_{n+3}^2+2(n^2+6n+22)R_{n+2}^2+2(n^2+10n+38)R_{n+1}^2-(4n^2+26n+103)R_{n+3}R_{n+2}-(4n^2+38n+137)R_{n+3}R_{n+1}+(4n^2+34n+125)R_{n+2}R_{n+1}-47)$.

Taking $R_n=C_n$ with $C_0=3, C_1=0, C_2=2$ in Theorem 3.1, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

COROLLARY 3.3. *For $n\geq 0$, Pell-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} kC_k^2 = \frac{1}{4}((2n^2+18n+69)C_{n+3}^2+(2n^2+14n+53)C_{n+2}^2+(2n^2+18n+85)C_{n+1}^2-4(n^2+8n+31)C_{n+3}C_{n+2}-4(n^2+10n+40)C_{n+3}C_{n+1}+4(n^2+10n+38)C_{n+2}C_{n+1}-89)$.

(b): $\sum_{k=0}^{n} kC_{k+1}C_k = \frac{1}{4}(-2(n^2+8n+31)C_{n+3}^2-2(n^2+6n+24)C_{n+2}^2-2(n^2+10n+40)C_{n+1}^2+(4n^2+30n+111)C_{n+3}C_{n+2}+(4n^2+38n+145)C_{n+3}C_{n+1}-(4n^2+38n+137)C_{n+2}C_{n+1}+84)$.

(c): $\sum_{k=0}^{n} kC_{k+2}C_k = \frac{1}{4}(2(n^2+8n+29)C_{n+3}^2+2(n^2+6n+22)C_{n+2}^2+2(n^2+10n+38)C_{n+1}^2-(4n^2+26n+103)C_{n+3}C_{n+2}-(4n^2+38n+137)C_{n+3}C_{n+1}+(4n^2+34n+125)C_{n+2}C_{n+1}-80)$.

If $x=1$, $r=1, s=1, t=2$ then we have the following theorem (in fact taking $r=1, s=1, t=2$ in Theorem 2.1 and then using L'Hospital rule twice for $x=1$ we obtain the following theorem).

THEOREM 3.4. *If $r=1, s=1, t=2$ then for $n\geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} kW_k^2 = \frac{1}{1323}((63n^2+198n-4076)W_{n+3}^2+9(21n^2+31n-1381)W_{n+2}^2+(252n^2-27n-16583)W_{n+1}^2-9(21n^2+45n-1366)W_{n+3}W_{n+2}-2(63n^2+135n-4070)W_{n+3}W_{n+1}+12(21n+19)W_{n+2}W_{n+1}+4211W_2^2+12519W_1^2+16304W_0^2-12510W_1W_2-8284W_2W_0+24W_1W_0)$.

(b): $\sum_{k=0}^{n} kW_{k+1}W_k = \frac{1}{2646}(-(63n^2+9n-4142)W_{n+3}^2 - 3(63n^2+51n-4174)W_{n+2}^2 - 4(63n^2+135n-4070)W_{n+1}^2 + 3(63n^2+93n-4192)W_{n+3}W_{n+2} + 2(63n^2+387n-3968)W_{n+3}W_{n+1} - 6(189n+73)W_{n+2}W_{n+1} - 4088W_2^2 - 12486W_1^2 - 16568W_0^2 + 12666W_2W_1 + 8584W_0W_2 - 696W_0W_1)$.

(c): $\sum_{k=0}^{n} kW_{k+2}W_k = \frac{1}{2646}(-(63n^2-117n-4130)W_{n+3}^2 - 9(21n^2+73n-1336)W_{n+2}^2 - 4(63n^2+9n-4184)W_{n+1}^2 + 27(7n^2+29n-452)W_{n+3}W_{n+2} + 2(63n^2-180n-4187)W_{n+3}W_{n+1} - 12(21n+40)W_{n+2}W_{n+1} - 3950W_2^2 - 12492W_1^2 - 16520W_0^2 + 12\,798W_2W_1 + 7888W_2W_0 + 228W_1W_0)$.

From Theorem 3.4, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

COROLLARY 3.5. *For $n \geq 0$, third order Jacobsthal numbers have the following properties:*

(a): $\sum_{k=0}^{n} kJ_k^2 = \frac{1}{1323}((63n^2+198n-4076)J_{n+3}^2 + 9(21n^2+31n-1381)J_{n+2}^2 + (252n^2-27n-16583)J_{n+1}^2 - 9(21n^2+45n-1366)J_{n+3}J_{n+2} - 2(63n^2+135n-4070)J_{n+3}J_{n+1} + 12(21n+19)J_{n+2}J_{n+1} + 4220)$.

(b): $\sum_{k=0}^{n} kJ_{k+1}J_k = \frac{1}{2646}(-(63n^2+9n-4142)J_{n+3}^2 - 3(63n^2+51n-4174)J_{n+2}^2 - 4(63n^2+135n-4070)J_{n+1}^2 + 3(63n^2+93n-4192)J_{n+3}J_{n+2} + 2(63n^2+387n-3968)J_{n+3}J_{n+1} - 6(189n+73)J_{n+2}J_{n+1} - 3908)$.

(c): $\sum_{k=0}^{n} kJ_{k+2}J_k = \frac{1}{2646}(-(63n^2-117n-4130)J_{n+3}^2 - 9(21n^2+73n-1336)J_{n+2}^2 - 4(63n^2+9n-4184)J_{n+1}^2 + 27(7n^2+29n-452)J_{n+3}J_{n+2} + 2(63n^2-180n-4187)J_{n+3}J_{n+1} - 12(21n+40)J_{n+2}J_{n+1} - 3644)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in Theorem 3.4, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

COROLLARY 3.6. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} kj_k^2 = \frac{1}{1323}((63n^2+198n-4076)j_{n+3}^2 + 9(21n^2+31n-1381)j_{n+2}^2 + (252n^2-27n-16583)j_{n+1}^2 - 9(21n^2+45n-1366)j_{n+3}j_{n+2} - 2(63n^2+135n-4070)j_{n+3}j_{n+1} + 12(21n+19)j_{n+2}j_{n+1} + 37668)$.

(b): $\sum_{k=0}^{n} kj_{k+1}j_k = \frac{1}{2646}(-(63n^2+9n-4142)j_{n+3}^2 - 3(63n^2+51n-4174)j_{n+2}^2 - 4(63n^2+135n-4070)j_{n+1}^2 + 3(63n^2+93n-4192)j_{n+3}j_{n+2} + 2(63n^2+387n-3968)j_{n+3}j_{n+1} - 6(189n+73)j_{n+2}j_{n+1} - 33180)$.

(c): $\sum_{k=0}^{n} kj_{k+2}j_k = \frac{1}{2646}(-(63n^2-117n-4130)j_{n+3}^2 - 9(21n^2+73n-1336)j_{n+2}^2 - 4(63n^2+9n-4184)j_{n+1}^2 + 27(7n^2+29n-452)j_{n+3}j_{n+2} + 2(63n^2-180n-4187)j_{n+3}j_{n+1} - 12(21n+40)j_{n+2}j_{n+1} - 33996)$.

3.2. The Case $x = -1$. We now consider the case $x = -1$ in Theorem 2.1.

Taking $x = -1, r = s = t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.7. *If $x = -1, r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{4}((-1)^n((n+1)W_{n+3}^2 - (2n+1)W_{n+2}^2 + (3n+2)W_{n+1}^2 - 2(n+2)W_{n+1}W_{n+3} + 2W_{n+2}W_{n+1}) + W_1^2 - W_0^2 - 2W_2W_0 + 2W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{4}((-1)^n((n+1)W_{n+3}^2 + W_{n+2}^2 - (n+2)W_{n+1}^2 - (2n+2)W_{n+3}W_{n+2} + 2nW_{n+2}W_{n+1}) + W_1^2 - W_0^2 - 2W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{4}((-1)^n(nW_{n+3}^2 - (2n+1)W_{n+2}^2 - (n+1)W_{n+1}^2 + 2W_{n+3}W_{n+2} + 2nW_{n+3}W_{n+1} - 4(n+1)W_{n+2}W_{n+1}) - W_2^2 + W_1^2 + 2W_2W_1 - 2W_2W_0)$.

From Proposition 3.7, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.8. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k T_k^2 = \frac{1}{4}((-1)^n\left((n+1)T_{n+3}^2 - (2n+1)T_{n+2}^2 + (3n+2)T_{n+1}^2 - 2(n+2)T_{n+1}T_{n+3} + 2T_{n+2}T_{n+1}\right) + 1)$.

(b): $\sum_{k=0}^{n} k(-1)^k T_{k+1}T_k = \frac{1}{4}((-1)^n\left((n+1)T_{n+3}^2 + T_{n+2}^2 - (n+2)T_{n+1}^2 - (2n+2)T_{n+3}T_{n+2} + 2nT_{n+2}T_{n+1}\right) + 1)$.

(c): $\sum_{k=0}^{n} k(-1)^k T_{k+2}T_k = \frac{1}{4}((-1)^n\left(nT_{n+3}^2 - (2n+1)T_{n+2}^2 - (n+1)T_{n+1}^2 + 2T_{n+3}T_{n+2} + 2nT_{n+3}T_{n+1} - 4(n+1)T_{n+2}T_{n+1}\right) + 2)$.

Taking $T_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in Proposition 3.7, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 3.9. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k K_k^2 = \frac{1}{4}((-1)^n\left((n+1)K_{n+3}^2 - (2n+1)K_{n+2}^2 + (3n+2)K_{n+1}^2 - 2(n+2)K_{n+1}K_{n+3} + 2K_{n+2}K_{n+1}\right) - 20)$.

(b): $\sum_{k=0}^{n} k(-1)^k K_{k+1}K_k = \frac{1}{4}((-1)^n\left((n+1)K_{n+3}^2 + K_{n+2}^2 - (n+2)K_{n+1}^2 - (2n+2)K_{n+3}K_{n+2} + 2nK_{n+2}K_{n+1}\right) - 14)$.

(c): $\sum_{k=0}^{n} k(-1)^k K_{k+2}K_k = \frac{1}{4}((-1)^n\left(nK_{n+3}^2 - (2n+1)K_{n+2}^2 - (n+1)K_{n+1}^2 + 2K_{n+3}K_{n+2} + 2nK_{n+3}K_{n+1} - 4(n+1)K_{n+2}K_{n+1}\right) - 20)$.

Taking $x = -1, r = 2, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

Proposition 3.10. *If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{75}((-1)^n\left((5n+2)W_{n+3}^2 - (45n+53)W_{n+2}^2 + (70n+68)W_{n+1}^2 + 2(5n+12)W_{n+3}W_{n+2} - 2(15n+31)W_{n+3}W_{n+1} + 2(10n+39)W_{n+2}W_{n+1}\right) - 3W_2^2 - 8W_1^2 - 2W_0^2 + 14W_2W_1 - 32W_0W_2 + 58W_1W_0)$. 14

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{75}((-1)^n\left((15n+16)W_{n+3}^2 + 3(5n+17)W_{n+2}^2 - (15n+31)W_{n+1}^2 - (45n+58)W_{n+3}W_{n+2} - (15n+21)W_{n+3}W_{n+1} + (60n+49)W_{n+2}W_{n+1}\right) + W_2^2 + 36W_1^2 - 16W_0^2 - 13W_1W_2 - 6W_2W_0 - 11W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{75}((-1)^n(20n+3)W_{n+3}^2 - (-1)^n(30n+17)W_{n+2}^2 - (-1)^n(20n+23)W_{n+1}^2 - (-1)^n(35n-11)W_{n+3}W_{n+2} + (-1)^n(30n+7)W_{n+3}W_{n+1} - (-1)^n(70n+83)W_{n+2}W_{n+1} - 17W_2^2 + 13W_1^2 - 3W_0^2 + 46W_1W_2 - 23W_2W_0 - 13W_1W_0)$.

From Proposition 3.10, we have the following Corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 3.11. *For $n \geq 0$, third-order Pell numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k P_k^2 = \frac{1}{75}((-1)^n\left((5n+2)P_{n+3}^2 - (45n+53)P_{n+2}^2 + (70n+68)P_{n+1}^2 + 2(5n+12)P_{n+3}P_{n+2} - 2(15n+31)P_{n+3}P_{n+1} + 2(10n+39)P_{n+2}P_{n+1}\right) + 8)$.

(b): $\sum_{k=0}^{n} k(-1)^k P_{k+1}P_k = \frac{1}{75}((-1)^n\left((15n+16)P_{n+3}^2 + 3(5n+17)P_{n+2}^2 - (15n+31)P_{n+1}^2 - (45n+58)P_{n+3}P_{n+2} - (15n+21)P_{n+3}P_{n+1} + (60n+49)P_{n+2}P_{n+1}\right) + 14)$.

(c): $\sum_{k=0}^{n} k(-1)^k P_{k+2}P_k = \frac{1}{75}((-1)^n(20n+3)P_{n+3}^2 - (-1)^n(30n+17)P_{n+2}^2 - (-1)^n(20n+23)P_{n+1}^2 - (-1)^n(35n-11)P_{n+3}P_{n+2} + (-1)^n(30n+7)P_{n+3}P_{n+1} - (-1)^n(70n+83)P_{n+2}P_{n+1} + 37)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in Proposition 3.10, we have the following Corollary which presents sum formulas of third-order Pell-Lucas numbers.

COROLLARY 3.12. *For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k Q_k^2 = \frac{1}{75}((-1)^n ((5n+2)Q_{n+3}^2 - (45n+53)Q_{n+2}^2 + (70n+68)Q_{n+1}^2 + 2(5n+12)Q_{n+3}Q_{n+2} - 2 (15n+31)Q_{n+3}Q_{n+1} + 2(10n+39)Q_{n+2}Q_{n+1}) - 218)$.

(b): $\sum_{k=0}^{n} k(-1)^k Q_{k+1}Q_k = \frac{1}{75}((-1)^n ((15n+16)Q_{n+3}^2 + 3(5n+17)Q_{n+2}^2 - (15n+31)Q_{n+1}^2 - (45n+58)Q_{n+3}Q_{n+2} - (15n+21)Q_{n+3}Q_{n+1} + (60n+49)Q_{n+2}Q_{n+1}) - 294)$.

(c): $\sum_{k=0}^{n} k(-1)^k Q_{k+2}Q_k = \frac{1}{75}((-1)^n (20n+3)Q_{n+3}^2 - (-1)^n (30n+17)Q_{n+2}^2 - (-1)^n (20n+23)Q_{n+1}^2 - (-1)^n (35n-11)Q_{n+3}Q_{n+2} + (-1)^n (30n+7)Q_{n+3}Q_{n+1} - (-1)^n (70n+83)Q_{n+2}Q_{n+1} - 527)$.

From Proposition 3.10, we have the following Corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

COROLLARY 3.13. *For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k E_k^2 = \frac{1}{75}((-1)^n ((5n+2)E_{n+3}^2 - (45n+53)E_{n+2}^2 + (70n+68)E_{n+1}^2 + 2(5n+12)E_{n+3}E_{n+2} - 2 (15n+31)E_{n+3}E_{n+1} + 2(10n+39)E_{n+2}E_{n+1}) + 3)$.

(b): $\sum_{k=0}^{n} k(-1)^k E_{k+1}E_k = \frac{1}{75}((-1)^n ((15n+16)E_{n+3}^2 + 3(5n+17)E_{n+2}^2 - (15n+31)E_{n+1}^2 - (45n+58)E_{n+3}E_{n+2} - (15n+21)E_{n+3}E_{n+1} + (60n+49)E_{n+2}E_{n+1}) + 24)$.

(c): $\sum_{k=0}^{n} k(-1)^k E_{k+2}E_k = \frac{1}{75}((-1)^n (20n+3)E_{n+3}^2 - (-1)^n (30n+17)E_{n+2}^2 - (-1)^n (20n+23)E_{n+1}^2 - (-1)^n (35n-11)E_{n+3}E_{n+2} + (-1)^n (30n+7)E_{n+3}E_{n+1} - (-1)^n (70n+83)E_{n+2}E_{n+1} + 42)$.

Taking $x = -1, r = 0, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.14. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{25}((-1)^n ((15n+14)W_{n+3}^2 - (15n-1)W_{n+2}^2 + (10n-4)W_{n+1}^2 + 2(5n+8)W_{n+3}W_{n+2} - 2 (10n+11)W_{n+2}W_{n+1} - 2(5n+13)W_{n+3}W_{n+1}) - W_2^2 + 16W_1^2 - 14W_0^2 + 6W_1W_2 - 16W_2W_0 - 2W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{25}((-1)^n ((5n+8)W_{n+3}^2 - (5n+3)W_{n+2}^2 - (5n+13)W_{n+1}^2 - (5n-2)W_{n+3}W_{n+2} + (10n-9)W_{n+2}W_{n+1} + (5n+3)W_{n+3}W_{n+1}) + 3W_2^2 + 2W_1^2 - 8W_0^2 + 7W_2W_1 - 2W_2W_0 - 19W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{25}((-1)^n ((10n+1)W_{n+3}^2 - (10n-9)W_{n+2}^2 - (10n+11)W_{n+1}^2 + (15n+19)W_{n+3}W_{n+2} + (10n-9)W_{n+3}W_{n+1} - (30n+23)W_{n+2}W_{n+1}) - 9W_2^2 + 19W_1^2 - W_0^2 + 4W_2W_1 - 19W_2W_0 + 7W_1W_0)$.

From Proposition 3.14, we have the following Corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

COROLLARY 3.15. *For $n \geq 0$, Padovan numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k P_k^2 = \frac{1}{25}((-1)^n ((15n+14)P_{n+3}^2 - (15n-1)P_{n+2}^2 + (10n-4)P_{n+1}^2 + 2(5n+8)P_{n+3}P_{n+2} - 2 (10n+11)P_{n+2}P_{n+1} - 2(5n+13)P_{n+3}P_{n+1}) - 11)$.

(b): $\sum_{k=0}^{n} k(-1)^k P_{k+1}P_k = \frac{1}{25}((-1)^n ((5n+8)P_{n+3}^2 - (5n+3)P_{n+2}^2 - (5n+13)P_{n+1}^2 - (5n-2)P_{n+3}P_{n+2} + (10n-9)P_{n+2}P_{n+1} + (5n+3)P_{n+3}P_{n+1}) - 17)$.

(c): $\sum_{k=0}^{n} k(-1)^k P_{k+2}P_k = \frac{1}{25}((-1)^n ((10n+1)P_{n+3}^2 - (10n-9)P_{n+2}^2 - (10n+11)P_{n+1}^2 + (15n+19)P_{n+3}P_{n+2} + (10n-9)P_{n+3}P_{n+1} - (30n+23)P_{n+2}P_{n+1}) + 1)$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in Proposition 3.14, we have the following Corollary which presents sum formulas of Perrin numbers.

COROLLARY 3.16. *For $n \geq 0$, Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k E_k^2 = \frac{1}{25}((-1)^n ((15n+14)E_{n+3}^2 - (15n-1)E_{n+2}^2 + (10n-4)E_{n+1}^2 + 2(5n+8)E_{n+3}E_{n+2} - 2(10n+11)E_{n+2}E_{n+1} - 2(5n+13)E_{n+3}E_{n+1}) - 226)$.

(b): $\sum_{k=0}^{n} k(-1)^k E_{k+1}E_k = \frac{1}{25}((-1)^n ((5n+8)E_{n+3}^2 - (5n+3)E_{n+2}^2 - (5n+13)E_{n+1}^2 - (5n-2)E_{n+3}E_{n+2} + (10n-9)E_{n+2}E_{n+1} + (5n+3)E_{n+3}E_{n+1}) - 72)$.

(c): $\sum_{k=0}^{n} k(-1)^k E_{k+2}E_k = \frac{1}{25}((-1)^n ((10n+1)E_{n+3}^2 - (10n-9)E_{n+2}^2 - (10n+11)E_{n+1}^2 + (15n+19)E_{n+3}E_{n+2} + (10n-9)E_{n+3}E_{n+1} - (30n+23)E_{n+2}E_{n+1}) - 159)$.

From Proposition 3.14, we have the following Corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

COROLLARY 3.17. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k S_k^2 = \frac{1}{25}((-1)^n ((15n+14)S_{n+3}^2 - (15n-1)S_{n+2}^2 + (10n-4)S_{n+1}^2 + 2(5n+8)S_{n+3}S_{n+2} - 2(10n+11)S_{n+2}S_{n+1} - 2(5n+13)S_{n+3}S_{n+1}) - 1)$.

(b): $\sum_{k=0}^{n} k(-1)^k S_{k+1}S_k = \frac{1}{25}((-1)^n ((5n+8)S_{n+3}^2 - (5n+3)S_{n+2}^2 - (5n+13)S_{n+1}^2 - (5n-2)S_{n+3}S_{n+2} + (10n-9)S_{n+2}S_{n+1} + (5n+3)S_{n+3}S_{n+1}) + 3)$.

(c): $\sum_{k=0}^{n} k(-1)^k S_{k+2}S_k = \frac{1}{25}((-1)^n ((10n+1)S_{n+3}^2 - (10n-9)S_{n+2}^2 - (10n+11)S_{n+1}^2 + (15n+19)S_{n+3}S_{n+2} + (10n-9)S_{n+3}S_{n+1} - (30n+23)S_{n+2}S_{n+1}) - 9)$.

Observe that setting $x = -1$, $r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 2.1 (a), (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = -1$, $r = 0, s = 2, t = 1$ then we have the following theorem (in fact taking $x = -1$, $r = 0, s = 2, t = 1$ in Theorem 2.1 and then using L'Hospital rule twice for $x = -1$ we obtain the following theorem).

THEOREM 3.18. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{100}((-1)^n ((20n^2 - 30n - 1569)W_{n+3}^2 - (20n^2 - 70n - 1519)W_{n+2}^2 - (20n^2 - 90n - 1679)W_{n+1}^2 + 4(5n^2 - 401)W_{n+3}W_{n+2} - 4(5n^2 + 10n - 396)W_{n+3}W_{n+1} - 4(15n^2 - 10n - 1198)W_{n+2}W_{n+1}) - 1519W_2^2 + 1429W_1^2 + 1569W_0^2 - 1584W_2W_1 + 1604W_2W_0 + 4692W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{100}((-1)^n (2(5n^2 - 401)W_{n+3}^2 - 2(5n^2 - 10n - 396)W_{n+2}^2 - 2(5n^2 + 10n - 396)W_{n+1}^2 + (10n^2 - 10n - 827)W_{n+3}W_{n+2} - (10n^2 + 10n - 827)W_{n+3}W_{n+1} - (30n^2 - 50n - 2461)W_{n+2}W_{n+1}) - 792W_2^2 + 762W_1^2 + 802W_0^2 - 807W_2W_1 + 827W_2W_0 + 2381W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{100}((-1)^n (2(15n^2 - 40n - 1173)W_{n+3}^2 - 2(15n^2 - 70n - 1118)W_{n+2}^2 - 2(15n^2 - 10n - 1198)W_{n+1}^2 + (30n^2 - 10n - 2381)W_{n+3}W_{n+2} - (30n^2 - 50n - 2461)W_{n+3}W_{n+1} - (90n^2 - 90n - 7103)W_{n+2}W_{n+1}) - 2236W_2^2 + 2066W_1^2 + 2346W_0^2 - 2341W_2W_1 + 2381W_2W_0 + 6923W_1W_0)$.

From Theorem 3.18, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $Q_0 = 1, R_1 = 1, R_2 = 1$).

COROLLARY 3.19. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k R_k^2 = \frac{1}{100}((-1)^n ((20n^2 - 30n - 1569)R_{n+3}^2 - (20n^2 - 70n - 1519)R_{n+2}^2 - (20n^2 - 90n - 1679)R_{n+1}^2 + 4(5n^2 - 401)R_{n+3}R_{n+2} - 4(5n^2 + 10n - 396)R_{n+3}R_{n+1} - 4(15n^2 - 10n - 1198)R_{n+2}R_{n+1}) + 6191)$.

(b): $\sum_{k=0}^{n} k(-1)^k R_{k+1}R_k = \frac{1}{100}((-1)^n\,(2(5n^2-401)R_{n+3}^2-2(5n^2-10n-396)R_{n+2}^2-2(5n^2+10n-396)R_{n+1}^2+(10n^2-10n-827)R_{n+3}R_{n+2}-(10n^2+10n-827)R_{n+3}R_{n+1}-(30n^2-50n-2461)R_{n+2}R_{n+1})+3173)$.

(c): $\sum_{k=0}^{n} k(-1)^k R_{k+2}R_k = \frac{1}{100}((-1)^n\,(2(15n^2-40n-1173)R_{n+3}^2-2(15n^2-70n-1118)R_{n+2}^2-2(15n^2-10n-1198)R_{n+1}^2+(30n^2-10n-2381)R_{n+3}R_{n+2}-(30n^2-50n-2461)R_{n+3}R_{n+1}-(90n^2-90n-7103)R_{n+2}R_{n+1})+9139)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in Theorem 3.18, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

COROLLARY 3.20. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k C_k^2 = \frac{1}{100}((-1)^n\,((20n^2-30n-1569)C_{n+3}^2-(20n^2-70n-1519)C_{n+2}^2-(20n^2-90n-1679)C_{n+1}^2+4(5n^2-401)C_{n+3}C_{n+2}-4(5n^2+10n-396)C_{n+3}C_{n+1}-4(15n^2-10n-1198)C_{n+2}C_{n+1})+17669)$.

(b): $\sum_{k=0}^{n} k(-1)^k C_{k+1}C_k = \frac{1}{100}((-1)^n\,(2(5n^2-401)C_{n+3}^2-2(5n^2-10n-396)C_{n+2}^2-2(5n^2+10n-396)C_{n+1}^2+(10n^2-10n-827)C_{n+3}C_{n+2}-(10n^2+10n-827)C_{n+3}C_{n+1}-(30n^2-50n-2461)C_{n+2}C_{n+1})+9012)$.

(c): $\sum_{k=0}^{n} k(-1)^k C_{k+2}C_k = \frac{1}{100}((-1)^n\,(2(15n^2-40n-1173)C_{n+3}^2-2(15n^2-70n-1118)C_{n+2}^2-2(15n^2-10n-1198)C_{n+1}^2+(30n^2-10n-2381)C_{n+3}C_{n+2}-(30n^2-50n-2461)C_{n+3}C_{n+1}-(90n^2-90n-7103)C_{n+2}C_{n+1})+26456)$.

Taking $x = -1, r = 0, s = 1, t = 2$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 3.21. *If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{64}((-1)^n\,((12n+5)\,W_{n+3}^2-(12n-7)W_{n+2}^2+(16n-4)W_{n+1}^2+2(4n+1)W_{n+3}W_{n+2}-4(4n+5)W_{n+3}W_{n+1}-4(4n-1)W_{n+2}W_{n+1})-7W_2^2+19W_1^2-20W_0^2-6W_2W_1-4W_2W_0+20W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{64}((-1)^n\,((4n+1)\,W_{n+3}^2-(4n-3)W_{n+2}^2-4(4n+5)W_{n+1}^2-(8n-2)W_{n+3}W_{n+2}+2(8n+6)W_{n+3}W_{n+1}+(16n-12)W_{n+2}W_{n+1})-3W_2^2+7W_1^2-4W_0^2+10W_2W_1-4W_2W_0-28W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{64}((-1)^n\,((4n-5)\,W_{n+3}^2-(4n-9)W_{n+2}^2-4(4n-1)W_{n+1}^2+(24n+14)W_{n+3}W_{n+2}+(16n-12)W_{n+3}W_{n+1}-2(24n+2)W_{n+2}W_{n+1})-9W_2^2+13W_1^2+20W_0^2-10W_2W_1-28W_2W_0+44W_1W_0)$.

From Proposition 3.21, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

COROLLARY 3.22. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k Q_k^2 = \frac{1}{64}((-1)^n\,((12n+5)\,Q_{n+3}^2-(12n-7)Q_{n+2}^2+(16n-4)Q_{n+1}^2+2(4n+1)Q_{n+3}Q_{n+2}-4(4n+5)Q_{n+3}Q_{n+1}-4(4n-1)Q_{n+2}Q_{n+1})+2)$.

(b): $\sum_{k=0}^{n} k(-1)^k Q_{k+1}Q_k = \frac{1}{64}((-1)^n\,((4n+1)\,Q_{n+3}^2-(4n-3)Q_{n+2}^2-4(4n+5)Q_{n+1}^2-(8n-2)Q_{n+3}Q_{n+2}+2(8n+6)Q_{n+3}Q_{n+1}+(16n-12)Q_{n+2}Q_{n+1})-22)$.

(c): $\sum_{k=0}^{n} k(-1)^k Q_{k+2}Q_k = \frac{1}{64}((-1)^n\,((4n-5)\,Q_{n+3}^2-(4n-9)Q_{n+2}^2-4(4n-1)Q_{n+1}^2+(24n+14)Q_{n+3}Q_{n+2}+(16n-12)Q_{n+3}Q_{n+1}-2(24n+2)Q_{n+2}Q_{n+1})+30)$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in Proposition 3.21, we have the following Corollary which presents sum formulas of Jacobsthal-Perrin numbers.

COROLLARY 3.23. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k L_k^2 = \frac{1}{64}((-1)^n((12n+5)L_{n+3}^2 - (12n-7)L_{n+2}^2 + (16n-4)L_{n+1}^2 + 2(4n+1)L_{n+3}L_{n+2} - 4(4n+5)L_{n+3}L_{n+1} - 4(4n-1)L_{n+2}L_{n+1}) - 232)$.

(b): $\sum_{k=0}^{n} k(-1)^k L_{k+1}L_k = \frac{1}{64}((-1)^n((4n+1)L_{n+3}^2 - (4n-3)L_{n+2}^2 - 4(4n+5)L_{n+1}^2 - (8n-2)L_{n+3}L_{n+2} + 2(8n+6)L_{n+3}L_{n+1} + (16n-12)L_{n+2}L_{n+1}) - 72)$.

(c): $\sum_{k=0}^{n} k(-1)^k L_{k+2}L_k = \frac{1}{64}((-1)^n((4n-5)L_{n+3}^2 - (4n-9)L_{n+2}^2 - 4(4n-1)L_{n+1}^2 + (24n+14)L_{n+3}L_{n+2} + (16n-12)L_{n+3}L_{n+1} - 2(24n+2)L_{n+2}L_{n+1}) - 24)$.

Taking $x=-1, r=1, s=0, t=1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.24. *If $r=1, s=0, t=1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{9}((-1)^n((3n+7)W_{n+3}^2 - (6n+5)W_{n+2}^2 + (6n-1)W_{n+1}^2 - 6W_{n+3}W_{n+2} - 2(3n+7)W_{n+3}W_{n+1} + 2(3n+10)W_{n+2}W_{n+1}) + 4W_2^2 + W_1^2 - 7W_0^2 - 6W_2W_1 - 8W_2W_0 + 14W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{9}((-1)^n((3n+4)W_{n+3}^2 + (3n+10)W_{n+2}^2 - (3n+7)W_{n+1}^2 - (9n+15)W_{n+3}W_{n+2} + (3n+10)W_{n+3}W_{n+1} + (6n-4)W_{n+2}W_{n+1}) + W_2^2 + 7W_1^2 - 4W_0^2 - 6W_2W_1 + 7W_2W_0 - 10W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{9}((-1)^n(-3W_{n+3}^2 - 3W_{n+2}^2 + 3W_{n+1}^2 + 9W_{n+3}W_{n+2} + (9n+6)W_{n+3}W_{n+1} - (9n+15)W_{n+2}W_{n+1}) - 3W_2^2 - 3W_1^2 + 3W_0^2 + 9W_2W_1 - 3W_2W_0 - 6W_1W_0)$.

From Proposition 3.24, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

COROLLARY 3.25. *For $n \geq 0$, Narayana numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k N_k^2 = \frac{1}{9}((-1)^n((3n+7)N_{n+3}^2 - (6n+5)N_{n+2}^2 + (6n-1)N_{n+1}^2 - 6N_{n+3}N_{n+2} - 2(3n+7)N_{n+3}N_{n+1} + 2(3n+10)N_{n+2}N_{n+1}) - 1)$.

(b): $\sum_{k=0}^{n} k(-1)^k N_{k+1}N_k = \frac{1}{9}((-1)^n((3n+4)N_{n+3}^2 + (3n+10)N_{n+2}^2 - (3n+7)N_{n+1}^2 - (9n+15)N_{n+3}N_{n+2} + (3n+10)N_{n+3}N_{n+1} + (6n-4)N_{n+2}N_{n+1}) + 2)$.

(c): $\sum_{k=0}^{n} k(-1)^k N_{k+2}N_k = \frac{1}{9}((-1)^n(-3N_{n+3}^2 - 3N_{n+2}^2 + 3N_{n+1}^2 + 9N_{n+3}N_{n+2} + (9n+6)N_{n+3}N_{n+1} - (9n+15)N_{n+2}N_{n+1}) + 3)$.

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in Proposition 3.24, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

COROLLARY 3.26. *For $n \geq 0$, Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k U_k^2 = \frac{1}{9}((-1)^n((3n+7)U_{n+3}^2 - (6n+5)U_{n+2}^2 + (6n-1)U_{n+1}^2 - 6U_{n+3}U_{n+2} - 2(3n+7)U_{n+3}U_{n+1} + 2(3n+10)U_{n+2}U_{n+1}) - 46)$.

(b): $\sum_{k=0}^{n} k(-1)^k U_{k+1}U_k = \frac{1}{9}((-1)^n((3n+4)U_{n+3}^2 + (3n+10)U_{n+2}^2 - (3n+7)U_{n+1}^2 - (9n+15)U_{n+3}U_{n+2} + (3n+10)U_{n+3}U_{n+1} + (6n-4)U_{n+2}U_{n+1}) - 43)$.

(c): $\sum_{k=0}^{n} k(-1)^k U_{k+2}U_k = \frac{1}{9}((-1)^n(-3U_{n+3}^2 - 3U_{n+2}^2 + 3U_{n+1}^2 + 9U_{n+3}U_{n+2} + (9n+6)U_{n+3}U_{n+1} - (9n+15)U_{n+2}U_{n+1}) + 3)$.

From Proposition 3.24, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

COROLLARY 3.27. *For $n \geq 0$, Narayana-Perrin numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k H_k^2 = \frac{1}{9}((-1)^n((3n+7)H_{n+3}^2 - (6n+5)H_{n+2}^2 + (6n-1)H_{n+1}^2 - 6H_{n+3}H_{n+2} - 2(3n+7) H_{n+3}H_{n+1} + 2(3n+10)H_{n+2}H_{n+1}) - 95)$.

(b): $\sum_{k=0}^{n} k(-1)^k H_{k+1}H_k = \frac{1}{9}((-1)^n((3n+4)H_{n+3}^2 + (3n+10)H_{n+2}^2 - (3n+7)H_{n+1}^2 - (9n+15)H_{n+3}H_{n+2} + (3n+10)H_{n+3}H_{n+1} + (6n-4)H_{n+2}H_{n+1}) + 10)$.

(c): $\sum_{k=0}^{n} k(-1)^k H_{k+2}H_k = \frac{1}{9}((-1)^n(-3H_{n+3}^2 - 3H_{n+2}^2 + 3H_{n+1}^2 + 9H_{n+3}H_{n+2} + (9n+6)H_{n+3}H_{n+1} - (9n+15) H_{n+2}H_{n+1}) - 3)$.

Taking $x=-1, r=1, s=1, t=2$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 3.28. *If $r=1, s=1, t=2$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{150}((-1)^n((20n+19)W_{n+3}^2 - (30n-19)W_{n+2}^2 + (70n-6)W_{n+1}^2 - (10n+37)W_{n+3}W_{n+2} - 2(30n+31)W_{n+3}W_{n+1} + 4(10n+22)W_{n+2}W_{n+1}) - W_2^2 + 49W_1^2 - 76W_0^2 - 27W_2W_1 - 2W_2W_0 + 48W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{300}((-1)^n((30n+1)W_{n+3}^2 + (30n+51)W_{n+2}^2 - 4(30n+31)W_{n+1}^2 - (90n+23)W_{n+3}W_{n+2} + 2(30n+51)W_{n+3}W_{n+1} + (60n-148)W_{n+2}W_{n+1}) - 29W_2^2 + 21W_1^2 - 4W_0^2 + 67W_2W_1 + 42W_2W_0 - 208W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{900}((-1)^n((30n-69)W_{n+3}^2 - (270n+219)W_{n+2}^2 - 4(30n-39)W_{n+1}^2 + (210n+387)W_{n+3}W_{n+2} + (360n-138)W_{n+3}W_{n+1} - 2(420n+144)W_{n+2}W_{n+1}) - 99W_2^2 + 51W_1^2 + 276W_0^2 + 177 W_2W_1 - 498W_2W_0 + 552W_1W_0)$.

From Proposition 3.28, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

COROLLARY 3.29. *For $n \geq 0$, third order Jacobsthal numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k J_k^2 = \frac{1}{150}((-1)^n((20n+19)J_{n+3}^2 - (30n-19)J_{n+2}^2 + (70n-6)J_{n+1}^2 - (10n+37)J_{n+3}J_{n+2} - 2 (30n+31)J_{n+3}J_{n+1} + 4(10n+22)J_{n+2}J_{n+1}) + 21)$.

(b): $\sum_{k=0}^{n} k(-1)^k J_{k+1}J_k = \frac{1}{300}((-1)^n((30n+1)J_{n+3}^2 + (30n+51)J_{n+2}^2 - 4(30n+31)J_{n+1}^2 - (90n+23)J_{n+3}J_{n+2} + 2(30n+51)J_{n+3}J_{n+1} + (60n-148)J_{n+2}J_{n+1}) + 59)$.

(c): $\sum_{k=0}^{n} k(-1)^k J_{k+2}J_k = \frac{1}{900}((-1)^n((30n-69)J_{n+3}^2 - (270n+219)J_{n+2}^2 - 4(30n-39)J_{n+1}^2 + (210n+387)J_{n+3}J_{n+2} + (360n-138)J_{n+3}J_{n+1} - 2(420n+144)J_{n+2}J_{n+1}) + 129)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in Proposition 3.28, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

COROLLARY 3.30. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k j_k^2 = \frac{1}{150}((-1)^n((20n+19)j_{n+3}^2 - (30n-19)j_{n+2}^2 + (70n-6)j_{n+1}^2 - (10n+37)j_{n+3}j_{n+2} - 2 (30n+31)j_{n+3}j_{n+1} + 4(10n+22)j_{n+2}j_{n+1}) - 339)$.

(b): $\sum_{k=0}^{n} k(-1)^k j_{k+1}j_k = \frac{1}{300}((-1)^n((30n+1)j_{n+3}^2 + (30n+51)j_{n+2}^2 - 4(30n+31)j_{n+1}^2 - (90n+23)j_{n+3}j_{n+2} + 2(30n+51)j_{n+3}j_{n+1} + (60n-148)j_{n+2}j_{n+1}) - 381)$.

(c): $\sum_{k=0}^{n} k(-1)^k j_{k+2}j_k = \frac{1}{900}((-1)^n((30n-69)j_{n+3}^2 - (270n+219)j_{n+2}^2 - 4(30n-39)j_{n+1}^2 + (210n+387)j_{n+3}j_{n+2} + (360n-138)j_{n+3}j_{n+1} - 2(420n+144)j_{n+2}j_{n+1}) - 4311)$.

Taking $x=-1, r=2, s=3, t=5$ in Theorem 2.1, we obtain the following Proposition.

Proposition 3.31. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{680625}((-1)^n ((15675n + 11674) W_{n+3}^2 - (9075n - 81169) W_{n+2}^2 + (288750n - 3100) W_{n+1}^2 - 2(17325n + 30966) W_{n+3}W_{n+2} - 10(14025n + 6407) W_{n+3}W_{n+1} + 10(23100n + 20113) W_{n+2}W_{n+1}) - 4001W_2^2 + 90244W_1^2 - 291850W_0^2 - 27282W_2W_1 + 76180W_2W_0 - 29870W_1W_0)$.

(b): $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{680625}((-1)^n ((14025n - 7618) W_{n+3}^2 + 11(5775n + 697) W_{n+2}^2 - 25(14025n + 6407) W_{n+1}^2 - (66825n - 22224) W_{n+3}W_{n+2} + 5(10725n + 18973) W_{n+3}W_{n+1} - (8250n + 254035) W_{n+2}W_{n+1}) - 21643W_2^2 - 55858W_1^2 + 190450W_0^2 + 89049W_2W_1 + 41240W_2W_0 - 245785W_1W_0)$.

(c): $\sum_{k=0}^{n} k(-1)^k W_{k+2}W_k = \frac{1}{75625}((-1)^n (550n - 1361) W_{n+3}^2 - 11(-1)^n (2200n + 981) W_{n+2}^2 - 25(-1)^n (550n - 811) W_{n+1}^2 - 5(-1)^n (14\,300n - 1011) W_{n+2}W_{n+1} + 5(-1)^n (2200n - 1429) W_{n+3}W_{n+1} + (-1)^n (10\,725n + 9073) W_{n+3}W_{n+2} - 1911W_2^2 + 13\,409W_1^2 + 34\,025W_0^2 - 1652W_2W_1 - 18\,145W_2W_0 + 76\,555W_1W_0)$.

From Proposition 3.31, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 3.32. *For $n \geq 0$, 3-primes numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k G_k^2 = \frac{1}{680625}((-1)^n ((15675n + 11674) G_{n+3}^2 - (9075n - 81169) G_{n+2}^2 + (288750n - 3100) G_{n+1}^2 - 2(17325n + 30966) G_{n+3}G_{n+2} - 10(14025n + 6407) G_{n+3}G_{n+1} + 10(23100n + 20113) G_{n+2}G_{n+1}) + 19676)$.

(b): $\sum_{k=0}^{n} k(-1)^k G_{k+1}G_k = \frac{1}{680625}((-1)^n ((14025n - 7618) G_{n+3}^2 + 11(5775n + 697) G_{n+2}^2 - 25(14025n + 6407) G_{n+1}^2 - (66825n - 22224) G_{n+3}G_{n+2} + 5(10725n + 18973) G_{n+3}G_{n+1} - (8250n + 254035) G_{n+2}G_{n+1}) + 35668)$.

(c): $\sum_{k=0}^{n} k(-1)^k G_{k+2}G_k = \frac{1}{75625}((-1)^n (550n - 1361) G_{n+3}^2 - 11(-1)^n (2200n + 981) G_{n+2}^2 - 25(-1)^n (550n - 811) G_{n+1}^2 - 5(-1)^n (14\,300n - 1011) G_{n+2}G_{n+1} + 5(-1)^n (2200n - 1429) G_{n+3}G_{n+1} + (-1)^n (10\,725n + 9073) G_{n+3}G_{n+2} + 2461)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in Proposition 3.31, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 3.33. *For $n \geq 0$, Lucas 3-primes numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k H_k^2 = \frac{1}{680625}((-1)^n ((15675n + 11674) H_{n+3}^2 - (9075n - 81169) H_{n+2}^2 + (288750n - 3100) H_{n+1}^2 - 2(17325n + 30966) H_{n+3}H_{n+2} - 10(14025n + 6407) H_{n+3}H_{n+1} + 10(23100n + 20113) H_{n+2}H_{n+1}) - 1105234)$.

(b): $\sum_{k=0}^{n} k(-1)^k H_{k+1}H_k = \frac{1}{680625}((-1)^n ((14025n - 7618) H_{n+3}^2 + 11(5775n + 697) H_{n+2}^2 - 25(14025n + 6407) H_{n+1}^2 - (66825n - 22224) H_{n+3}H_{n+2} + 5(10725n + 18973) H_{n+3}H_{n+1} - (8250n + 254035) H_{n+2}H_{n+1}) + 869788)$.

(c): $\sum_{k=0}^{n} k(-1)^k H_{k+2}H_k = \frac{1}{75625}((-1)^n (550n - 1361) H_{n+3}^2 - 11(-1)^n (2200n + 981) H_{n+2}^2 - 25(-1)^n (550n - 811) H_{n+1}^2 - 5(-1)^n (14\,300n - 1011) H_{n+2}H_{n+1} + 5(-1)^n (2200n - 1429) H_{n+3}H_{n+1} + (-1)^n (10\,725n + 9073) H_{n+3}H_{n+2} + 50701)$.

From Proposition 3.31, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 3.34. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

(a): $\sum_{k=0}^{n} k(-1)^k E_k^2 = \frac{1}{680625}((-1)^n ((15675n + 11674) E_{n+3}^2 - (9075n - 81169)E_{n+2}^2 + (288750n - 3100)E_{n+1}^2 - 2(17325n + 30966)E_{n+3}E_{n+2} - 10(14025n + 6407)E_{n+3}E_{n+1} + 10(23100n + 20113)E_{n+2}E_{n+1}) + 58961)$.

(b): $\sum_{k=0}^{n} k(-1)^k E_{k+1}E_k = \frac{1}{680625}((-1)^n ((14025n - 7618) E_{n+3}^2 + 11(5775n + 697)E_{n+2}^2 - 25(14025n + 6407)E_{n+1}^2 - (66825n - 22224)E_{n+3}E_{n+2} + 5(10725n + 18973)E_{n+3}E_{n+1} - (8250n + 254035)E_{n+2}E_{n+1}) + 11548)$.

(c): $\sum_{k=0}^{n} k(-1)^k E_{k+2}E_k = \frac{1}{75625}((-1)^n (550n - 1361) E_{n+3}^2 - 11(-1)^n (2200n + 981) E_{n+2}^2 - 25(-1)^n (550n - 811)E_{n+1}^2 - 5(-1)^n (14300n - 1011)E_{n+2}E_{n+1} + 5(-1)^n (2200n - 1429)E_{n+3}E_{n+1} + (-1)^n (10725n + 9073)E_{n+3}E_{n+2} + 9846)$.

3.3. The Case $x = i$. We now consider the complex case $x = i$ in Theorem 2.1. The following proposition presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Taking $x = i, r = s = t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.35. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} ki^k W_k^2 = \frac{-i}{4}(i^n (i((1+i)n + 2 + i) W_{n+3}^2 + (2in - 2 + 4i)W_{n+2}^2 - i((1+3i)n + 2 + i)W_{n+1}^2 + 2((1-i)n + 2 - 3i) W_{n+3}W_{n+2} + (2+2i)W_{n+3}W_{n+1} - 2((1+i)n + 4 + i)W_{n+2}W_{n+1}) - W_2^2 - (2+2i)W_1^2 + (1-2i)W_0^2 + (4+2i)W_2W_1 - 6iW_1W_0 - (2-2i)W_2W_0)$.

(b): $\sum_{k=0}^{n} ki^k W_{k+1}W_k = \frac{-i}{4}(i^n ((1-i)W_{n+3}^2 - 2(in + 1 + 2i)W_{n+2}^2 + (1+i)W_{n+1}^2 + (2in + 4i)W_{n+3}W_{n+2} + i(2in + 2 + 4i)W_{n+3}W_{n+1} - i((2+4i)n + 4 + 4i)W_{n+2}W_{n+1}) + (1+i)W_2^2 + (2-2i)W_1^2 - (1-i)W_0^2 - 2W_2W_1 - (2+2i)W_2W_0 + 2W_1W_0)$.

(c): $\sum_{k=0}^{n} ki^k W_{k+2}W_k = \frac{-i}{4}(i^n (i((1+i)n + 2 - i) W_{n+3}^2 + (2n + 2 - 2i)W_{n+2}^2 - ((1+i)n + 3)W_{n+1}^2 - (2+2i) W_{n+3}W_{n+2} - i((2+2i)n + 2)W_{n+3}W_{n+1} - (2-2i)W_{n+2}W_{n+1}) - (1-2i)W_2^2 + 2W_1^2 - (1+2i)W_0^2 + (2-2i)W_2W_1 - 2iW_2W_0 - (2+2i)W_1W_0)$.

From Proposition 3.35, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 3.36. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a): $\sum_{k=0}^{n} ki^k T_k^2 = \frac{-i}{4}(i^n (i((1+i)n + 2 + i) T_{n+3}^2 + (2in - 2 + 4i)T_{n+2}^2 - i((1+3i)n + 2 + i)T_{n+1}^2 + 2((1-i)n + 2 - 3i)T_{n+3}T_{n+2} + (2+2i)T_{n+3}T_{n+1} - 2((1+i)n + 4 + i)T_{n+2}T_{n+1}) + 1)$.

(b): $\sum_{k=0}^{n} ki^k T_{k+1}T_k = \frac{-i}{4}(i^n ((1-i)T_{n+3}^2 - 2(in + 1 + 2i)T_{n+2}^2 + (1+i)T_{n+1}^2 + (2in + 4i)T_{n+3}T_{n+2} + i(2in + 2 + 4i) T_{n+3}T_{n+1} - i((2+4i)n + 4 + 4i)T_{n+2}T_{n+1}) + 1 - i)$.

(c): $\sum_{k=0}^{n} ki^k T_{k+2}T_k = \frac{-i}{4}(i^n (i((1+i)n + 2 - i) T_{n+3}^2 + (2n + 2 - 2i)T_{n+2}^2 - ((1+i)n + 3)T_{n+1}^2 - (2+2i) T_{n+3}T_{n+2} - i((2+2i)n + 2)T_{n+3}T_{n+1} - (2-2i)T_{n+2}T_{n+1}) + 3)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in Proposition 3.35, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 3.37. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} ki^k K_k^2 = \frac{-i}{4}(i^n (i((1+i)n + 2 + i) K_{n+3}^2 + (2in - 2 + 4i)K_{n+2}^2 - i((1+3i)n + 2 + i)K_{n+1}^2 + 2((1-i)n + 2 - 3i)K_{n+3}K_{n+2} + (2+2i)K_{n+3}K_{n+1} - 2((1+i)n + 4 + i)K_{n+2}K_{n+1}) - 8 - 14i)$.

(b): $\sum_{k=0}^{n} ki^k K_{k+1}K_k = \frac{-i}{4}(i^n ((1-i)K_{n+3}^2 - 2(in + 1 + 2i)K_{n+2}^2 + (1+i)K_{n+1}^2 + (2in + 4i)K_{n+3}K_{n+2} + i(2in + 2 + 4i)K_{n+3}K_{n+1} - i((2+4i)n + 4 + 4i)K_{n+2}K_{n+1}) - 16 - 2i)$.

(c): $\sum_{k=0}^{n} ki^k K_{k+2}K_k = \frac{-i}{4}(i^n(i((1+i)n+2-i)K_{n+3}^2+(2n+2-2i)K_{n+2}^2-((1+i)n+3)K_{n+1}^2-(2+2i)$ $K_{n+3}K_{n+2}-i((2+2i)n+2)K_{n+3}K_{n+1}-(2-2i)K_{n+2}K_{n+1})-16-30i).$

Corresponding sums of the other third order linear sequences can be calculated similarly when $x=i$.

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